

# The formal theories of presheaves and cocompletions

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# Cocompletions

small category  $\mathbf{A}$  + small colimits =  $\mathcal{P}[\mathbf{A}]$

**FinSet** + filtered colimits = **Set**

small finitely complete category + filtered colimits = locally finitely presentable category

small category + filtered colimits = finitely accessible category

**FinSet** + sifted colimits = **Set**

finitary algebraic theory  $\mathcal{T}$  + sifted colimits =  $\mathcal{T}$ -algebras

category  $\mathbf{A}$  + absolute colimits = Cauchy completion of  $\mathbf{A}$

# Cocompletions in **Cat**

Let  $\Phi$  be a class of small categories.

- a category  $\mathbf{E}$  is  **$\Phi$ -cocomplete** if  $\mathbf{E}$  has all colimits  $\text{colim}_{i \in \mathbf{I}} f(i)$  with  $\mathbf{I} \in \Phi$
- a functor  $g: \mathbf{E} \rightarrow \mathbf{X}$  is  **$\Phi$ -cocontinuous** if  $g$  preserves these colimits

## Definition ( $\Phi$ -cocompletion)

The  **$\Phi$ -cocompletion** of a category  $\mathbf{A}$  is a functor  $\phi_{\mathbf{A}}: \mathbf{A} \rightarrow \Phi(\mathbf{A})$  such that:

1.  $\Phi(\mathbf{A})$  is  **$\Phi$ -cocomplete**.
2. For all  $f: \mathbf{A} \rightarrow \mathbf{X}$ , with  **$\Phi$ -cocomplete** codomain, there is an essentially unique  **$\Phi$ -cocontinuous**  $\tilde{f}: \Phi(\mathbf{A}) \rightarrow \mathbf{X}$  such that  $f \cong \tilde{f} \circ \phi_{\mathbf{A}}$ .

# Cocompletions in **Cat**

## *Example ( $\Phi = \text{filtered categories}$ )*

The *filtered cocompletion* of a category  $\mathbf{A}$  is a functor  $\phi_{\mathbf{A}}: \mathbf{A} \rightarrow \text{Ind}(\mathbf{A})$  such that:

1.  $\text{Ind}(\mathbf{A})$  has filtered colimits;
2. For all  $f: \mathbf{A} \rightarrow \mathbf{X}$ , such that  $\mathbf{X}$  has filtered colimits, there is an essentially unique finitary  $\tilde{f}: \text{Ind}(\mathbf{A}) \rightarrow \mathbf{X}$  such that  $f \cong \tilde{f} \circ \phi_{\mathbf{A}}$ .

The filtered cocompletion of **FinSet** is

$$\mathbf{FinSet} \hookrightarrow \mathbf{Set}$$

There is plenty of work on cocompletions in  $\mathbb{V}$ -**Cat**, where  $\mathbb{V}$  is a complete and cocomplete smcc.

# NOTES ON ENRICHED CATEGORIES WITH COLIMITS OF SOME CLASS

G.M. KELLY AND V. SCHMITT

# Virtual double categories

A virtual double category  $\mathbb{X}$  has

1. Objects  $X, A, \dots$

2. Composable *tight-cells*  $X \downarrow_f Y$

3. *Loose-cells*  $p: Y \rightarrow X$

4. Composable 2-cells

$$\begin{array}{ccccccc}
 X_0 & \xleftarrow{p_1} & X_1 & \xleftarrow{p_2} & \cdots & \xleftarrow{p_{n-1}} & X_{n-1} & \xleftarrow{p_n} & X_n \\
 f \downarrow & & & & & \Phi & & & \downarrow g \\
 A & \longleftarrow & & & & & & & B \\
 & & & & & & & & & & q
 \end{array}$$

Example:  $\mathbb{X} = \mathbb{V}\text{-Cat}$

1.  $\mathbb{V}$ -categories

2.  $\mathbb{V}$ -functors

3.  $\mathbb{V}$ -distributors, with  $p(x, y) \in \mathbb{V}$

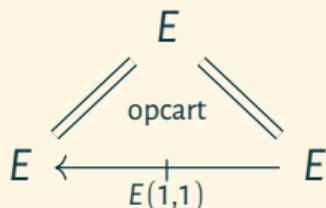
4.  $\mathbb{V}$ -natural transformations

$$p_1(x_0, x_1) \otimes \cdots \otimes p_n(x_{n-1}, x_n) \xrightarrow{\Phi_{x_0, \dots, x_n}} q(fx_0, gx_n)$$

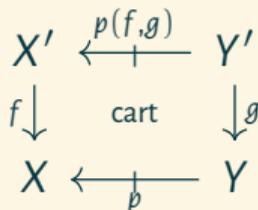
# Virtual equipments

A virtual equipment also has:

1. Loose-identities  $E(1, 1)$



2. Restrictions  $p(f, g)$



For  $\mathbb{X} = \mathbb{V}\text{-Cat}$ :

1.  $E(1, 1)(x, y) = E(x, y)$

2.  $p(f, g)(x, y) = p(fx, gy)$

Restrictions of  
loose-identities

$$E(f, g) = E(1, 1)(f, g)$$

Companions  
Conjoints

$$\left. \begin{array}{l} E(1_E, j): A \rightarrow E \\ E(j, 1_E): E \rightarrow A \end{array} \right\} j: A \rightarrow E$$

# Strict restrictions

$$\begin{array}{ccc} X' & \xleftarrow{p(f,g)} & Y' \\ f \downarrow & \text{cart} & \downarrow g \\ X & \xleftarrow{p} & Y \end{array}$$

We'll assume that restrictions are **chosen** and **strict**:

$$p(1_X, 1_Y) = p \quad p(f, g)(f', g') = p(ff', gg')$$

# Globular 2-cells

Since we have restrictions, instead of 2-cells

$$\begin{array}{ccccccc} X_0 & \xleftarrow{p_1} & X_1 & \xleftarrow{p_2} & \cdots & \xleftarrow{p_{n-1}} & X_{n-1} & \xleftarrow{p_n} & X_n \\ f \downarrow & & & & & \phi & & & \downarrow g \\ A & \xleftarrow{\quad} & & & & & & & B \\ & & & & & q & & & \end{array}$$

we can consider *globular* 2-cells

$$\psi: p_1, \dots, p_n \Rightarrow q(f, g)$$

$$\begin{array}{ccccccc} X_0 & \xleftarrow{p_1} & X_1 & \xleftarrow{p_2} & \cdots & \xleftarrow{p_{n-1}} & X_{n-1} & \xleftarrow{p_n} & X_n \\ \parallel & & & & & \psi & & & \parallel \\ X_0 & \xleftarrow{\quad} & & & & & & & X_n \\ & & & & & q(f, g) & & & \end{array}$$

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# Colimits in virtual equipments

## Right lift $q \blacktriangleleft p$

$$\begin{array}{ccc}
 X & \xleftarrow{q \blacktriangleleft p} & Y \\
 & \searrow p & \downarrow q \\
 & & A
 \end{array}
 \Rightarrow
 \begin{array}{c}
 \text{such that } \frac{r_1, \dots, r_n \Rightarrow q \blacktriangleleft p}{p, r_1, \dots, r_n \Rightarrow q}
 \end{array}$$

## Weighted colimit $p \circledast f$

$$\begin{array}{ccc}
 X & \xrightarrow{p \circledast f} & Y \\
 & \searrow p & \uparrow f \\
 & & A
 \end{array}
 \text{ such that } Y(p \circledast f, 1) \cong Y(f, 1) \blacktriangleleft p$$

For  $\mathbb{X} = \mathbb{V}\text{-Cat}$ :

$$(q \blacktriangleleft p)(x, y) \cong \mathcal{P}A(p(-, x), q(-, y))$$

$$\cong \int_{a \in A} p(a, x) \multimap q(a, y)$$

assuming  $\mathbb{V}$  is closed

$$Y((p \circledast f)x, y) \cong \mathcal{P}A(p(-, x), Y(f-, y))$$

# Left extensions in virtual equipments

## Weighted colimit $p \circledast f$

$$\begin{array}{ccc}
 X & \xrightarrow{p \circledast f} & Y \\
 & \searrow p & \uparrow f \\
 & & A
 \end{array}
 \text{ such that } Y(p \circledast f, 1) \cong Y(f, 1) \blacktriangleleft p$$

## (Pointwise) left (Kan) extension $j \triangleright f$

$$\begin{array}{ccc}
 E & \xrightarrow{j \triangleright f} & Y \\
 & \swarrow j & \uparrow f \\
 & & A
 \end{array}
 \text{ such that } j \triangleright f \cong E(j, 1) \circledast f$$

For  $\mathbb{X} = \mathbb{V}\text{-Cat}$ :

$$Y((p \circledast f)x, y) \cong \mathcal{P}A(p(-, x), Y(f-, y))$$

$$Y((j \triangleright f)x, y) \cong \mathcal{P}A(E(j-, x), Y(f-, y))$$

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1.  $\Phi(\mathbf{A})$  is  **$\Phi$ -cocomplete**.
2. For all  $f: \mathbf{A} \rightarrow \mathbf{X}$ , with  **$\Phi$ -cocomplete** codomain, there is an essentially unique  **$\Phi$ -cocontinuous**  $\tilde{f}: \Phi(\mathbf{A}) \rightarrow \mathbf{X}$  such that  $f \cong \tilde{f} \circ \phi_{\mathbf{A}}$ .

# Cocompletions in a virtual equipment (wrong)

Let  $\Phi$  be a class of loose-cells.

- an object  $E$  is  **$\Phi$ -cocomplete** if  $E$  admits all colimits  $p \circledast f$  with  $p \in \Phi$
- a tight-cell  $g: E \rightarrow X$  is  **$\Phi$ -cocontinuous** if  $g$  preserves these colimits

## Definition ( $\Phi$ -cocompletion)

The  **$\Phi$ -cocompletion** of an object  $A$  is a tight-cell  $\phi_A: A \rightarrow \Phi(A)$  such that:

1.  $\Phi(A)$  is  **$\Phi$ -cocomplete**.
2. For all  $f: A \rightarrow X$ , with  **$\Phi$ -cocomplete** codomain, there is an essentially unique  **$\Phi$ -cocontinuous**  $\tilde{f}: \Phi(A) \rightarrow X$  such that  $f \cong \tilde{f} \circ \phi_A$ .

# Cocompletions in a virtual equipment (still wrong)

A loose-cell  $q: X \rightarrow E$  is  **$\Phi$ -cocontinuous** if  $q(p \circledast f, 1) \cong q(f, 1) \triangleleft p$  for  $p \in \Phi$

## Definition ( $\Phi$ -cocompletion)

The  **$\Phi$ -cocompletion** of  $A$  is a tight-cell  $\phi_A: A \rightarrow \Phi(A)$  such that:

1.  $\Phi(A)$  is  **$\Phi$ -cocomplete**.
2. For all  $f: A \rightarrow X$ , with  **$\Phi$ -cocomplete** codomain, there is an essentially unique  **$\Phi$ -cocontinuous**  $\tilde{f}: \Phi(A) \rightarrow X$  such that  $f \cong \tilde{f} \circ \phi_A$ .
3. For all  $p: X \rightarrow A$ , there is an essentially unique  **$\Phi$ -cocontinuous**  $\tilde{p}: X \rightarrow \Phi(A)$  such that  $p \cong \tilde{p}(\phi_A, 1)$ .

# Cocompletions in a virtual equipment

A loose-cell  $q: X \rightarrow E$  is  **$\Phi$ -cocontinuous** if  $q(p \circledast f, 1) \cong q(f, 1) \triangleleft p$  for  $p \in \Phi$

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The  **$\Phi$ -cocompletion** of  $A$  is a tight-cell  $\phi_A: A \rightarrow \Phi(A)$  such that:

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3. For all  $p: X \rightarrow A$ , there is an essentially unique  **$\Phi$ -cocontinuous**  $\tilde{p}: X \rightarrow \Phi(A)$  such that  $p \cong \tilde{p}(\phi_A, 1)$ .
4.  $\tilde{p}$  is the right lift  $p \triangleleft \Phi(A)(\phi_A, 1)$ , and the canonical 2-cell  $\tilde{p}(\phi_A, 1) \Rightarrow p$  is an isomorphism.

# $\phi_A$ is fully faithful

For every  $p: X \rightarrow A$ ,

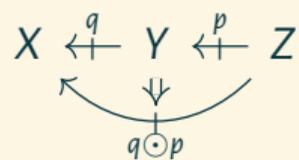
$$\begin{aligned} p \triangleleft \Phi(A)(\phi_A, \phi_A) &\cong (p \triangleleft \Phi(A)(\phi_A, 1))(\phi_A, 1) \\ &\cong \tilde{p}(\phi_A, 1) \\ &\cong p \\ &\cong p \triangleleft A(1, 1) \end{aligned}$$

So there is a bijection  $\frac{A(1, 1) \Rightarrow p}{\Phi(A)(\phi_A, \phi_A) \Rightarrow p}$  natural in  $p$ .

Hence  $A(1, 1) \cong \Phi(A)(\phi_A, \phi_A)$ .

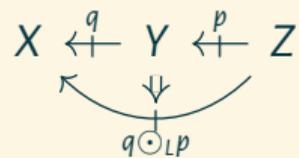
# Loose-composites in virtual equipments

## Composite $q \odot p$



such that 
$$\frac{r'_1, \dots, r'_m, q \odot p, r_1, \dots, r_n \Rightarrow s}{r'_1, \dots, r'_m, q, p, r_1, \dots, r_n \Rightarrow s}$$

## Left-composite $q \odot_L p$



such that 
$$\frac{q \odot_L p, r_1, \dots, r_n \Rightarrow s}{q, p, r_1, \dots, r_n \Rightarrow s}$$

# Absolute colimits

For  $j: A \rightarrow E$ , a colimit  $q \otimes f$  in  $E$  is *j-absolute* if

$$E(j, q \otimes f) \cong E(j, f) \odot_L q$$

If  $q \otimes f$  is *j-absolute*, then  $q \otimes f$  is preserved by every  $j \triangleright g$

# Absolute colimits

For  $j: A \rightarrow E$ , a colimit  $q \circledast f$  in  $E$  is  $j$ -absolute if

$$E(j, q \circledast f) \cong E(j, f) \odot_L q$$

$\Phi$ -colimits in  $\Phi(A)$  are  $\phi_A$ -absolute:

$$\begin{array}{c} \Phi(A)(\phi_A, q \circledast f), r_1, \dots, r_n \Rightarrow s \\ \hline r_1, \dots, r_n \Rightarrow s \blacktriangleleft \Phi(A)(\phi_A, q \circledast f) \\ \hline r_1, \dots, r_n \Rightarrow (s \blacktriangleleft \Phi(A)(\phi_A, 1))(q \circledast f, 1) \\ \hline r_1, \dots, r_n \Rightarrow (s \blacktriangleleft \Phi(A)(\phi_A, f)) \blacktriangleleft q \\ \hline q, r_1, \dots, r_n \Rightarrow s \blacktriangleleft \Phi(A)(\phi_A, f) \\ \hline \Phi(A)(\phi_A, f), q, r_1, \dots, r_n \Rightarrow s \end{array}$$

# $\Phi$ -cocontinuous tight-cells

The following are equivalent for a tight-cell  $g: \Phi(A) \rightarrow X$ :

1.  $g$  is  $\Phi$ -cocontinuous;
2.  $g$  is a left extension  $\phi_A \triangleright f$ , for some  $f: A \rightarrow X$ ;
3.  $g$  preserves  $\phi_A$ -absolute colimits.

# Presheaves in virtual equipments

## Definition (Presheaf object)

The presheaf object for  $A$  is a loose-cell  $\pi_A: \mathcal{P}[A] \rightarrow A$  such that:

1. For all  $q: Y \rightarrow A$ , there is a unique tight-cell  $\check{q}: Y \rightarrow \mathcal{P}[A]$  such that  $\pi_A(1, \check{q}) = q$ .
2.  $\pi_A$  is dense ( $\pi_A \blacktriangleleft \pi_A \cong \mathcal{P}[A](1, 1)$ ).

For  $\mathbb{X} = \mathbb{V}\text{-Cat}$ :

- Objects of  $\mathcal{P}[A]$  are  $\mathbb{V}$ -presheaves  $\phi$  on  $A$
- $\mathcal{P}[A](\phi, \psi) \cong \int_{a \in A} \phi(a) \multimap \psi(a)$   
assuming  $\mathbb{V}$  is closed
- $\pi_A(a, \phi) = \phi(a)$

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For  $\mathbb{X} = \mathbb{V}\text{-Cat}$ :

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assuming  $\mathbb{V}$  is closed
- $\pi_A(a, \phi) = \phi(a)$

- Right lifts:  $q \blacktriangleleft p \cong \mathcal{P}[A](\check{p}, \check{q})$

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- $\mathcal{P}[A](\phi, \psi) \cong \int_{a \in A} \phi(a) \multimap \psi(a)$   
assuming  $\mathbb{V}$  is closed
- $\pi_A(a, \phi) = \phi(a)$

• Right lifts:  $q \blacktriangleleft p \cong \mathcal{P}[A](\check{p}, \check{q})$

• Yoneda lemma:  $\mathcal{P}[A](\mathcal{Y}_A, \check{q}) \cong q$ , where  $\mathcal{Y}_A = \overline{A(1, 1)}: A \rightarrow \mathcal{P}[A]$

# Presheaves in virtual equipments

## Definition ( $\Phi$ -presheaf object)

The  $\Phi$ -presheaf object for  $A$  is a loose-cell  $\pi_A: \Phi[A] \rightarrow A$  such that:

1. For all  $q: Y \rightarrow A$  with  $q \in \Phi$ , there is a unique tight-cell  $\check{q}: Y \rightarrow \Phi[A]$  such that  $\pi_A(1, \check{q}) = q$ .
  2.  $\pi_A$  is dense ( $\pi_A \blacktriangleleft \pi_A \cong \Phi[A](1, 1)$ ).
  3.  $\pi_A(1, f) \in \Phi$ , for all  $f: Y \rightarrow \Phi[A]$ .
- Right lifts:  $q \blacktriangleleft p \cong \Phi[A](\check{p}, \check{q})$ , assuming  $p, q \in \Phi$
  - Yoneda lemma:  $\Phi[A](\phi_A, \check{q}) \cong q$ , where  $\phi_A = \overline{A(1, 1)}: A \rightarrow \Phi[A]$ , assuming  $A(1, 1) \in \Phi$  and  $q \in \Phi$

# Cocompletions are almost presheaf objects

Let  $\phi_A: A \rightarrow \Phi(A)$  be a  $\Phi$ -cocompletion, and assume every  $\tilde{p} \cong p \blacktriangleleft \Phi(A)(\phi_A, 1)$  is representable:

$$\tilde{p} \cong \Phi(A)(1, \check{p})$$

Then  $\check{p}$  is essentially unique such that  $\pi_A(1, \check{p}) \cong p$ , where  $\pi_A \cong \Phi(A)(\phi_A, 1)$

$$\begin{array}{ll} p: X \twoheadrightarrow A & \mapsto \check{p}: X \rightarrow \Phi(A) \\ f: X \rightarrow \Phi(A) & \mapsto \Phi(A)(\phi_A, f): X \twoheadrightarrow A \end{array}$$

# Colimits in presheaf objects

The colimit

$$\begin{array}{ccc} X & \xrightarrow{q \circledast f} & \Phi[A] \\ & \searrow q & \uparrow f \\ & & A \end{array}$$

is given by a left-composite

$$q \circledast f \cong \overbrace{\pi_A(1, f) \odot_L q}$$

assuming  $\pi_A(1, f) \odot_L q$  exists and is in  $\Phi$ .

This is  $\phi_A$ -absolute:

$$\Phi[A](\phi_A, q \circledast f) \cong \Phi[A](\phi_A, f) \odot_L q$$

# Cocompletions from presheaves

Assuming:

1. The presheaf object  $\pi_A: \Phi[A] \rightarrow A$  exists.
2.  $A(1, 1) \in \Phi$  (so we have  $\phi_A = \overline{A(1, 1)}: A \rightarrow \Phi[A]$ ).
3.  $\Phi$  is closed under left-composites (so  $\Phi[A]$  has  $\Phi$ -colimits).
4.  $q \blacktriangleleft p$  exists for all  $p \in \Phi$  (in particular, for  $p \cong \Phi[A](\phi_A, 1)$ ).

The  $\Phi$ -cocompletion of  $A$  is  $\phi_A: A \rightarrow \Phi[A]$ .

# Cocompletions from presheaves

Given  $\Psi$ , if there exists some  $\Phi$  such that:

1. The presheaf object  $\pi_A: \Phi[A] \rightarrow A$  exists.
2.  $A(1, 1) \in \Phi$  (so we have  $\phi_A = \overline{A(1, 1)}: A \rightarrow \Phi[A]$ ).
3.  $\Phi$  is closed under left-composites (so  $\Phi[A]$  has  $\Phi$ -colimits).
4.  $q \blacktriangleleft p$  exists for all  $p \in \Phi$  (in particular, for  $p \cong \Phi[A](\phi_A, 1)$ ).
5.  $\Psi$  and  $\Phi$  are colimit-equivalent.

The  $\Psi$ -cocompletion of  $A$  is  $\phi_A: A \rightarrow \Phi[A]$ .

# Cocompletions from presheaves, in $\mathbb{V}$ -Cat

If  $A$  is small, each  $p \in \Phi$  has small codomain, and the  $\mathbb{V}$ -category  $\mathcal{P}[A]$  exists:

$\Phi(A) \hookrightarrow \mathcal{P}[A]$  is the smallest full sub- $\mathbb{V}$ -category such that

- $\Phi(A)$  contains the representables
- $\Phi(A)$  is closed under  $\Phi$ -colimits

$\phi_A: A \rightarrow \Phi(A)$  is the Yoneda embedding

# Cocompletions in a virtual equipment

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3. For all  $p: X \rightarrow A$ , there is an essentially unique  $\Phi$ -cocontinuous  $\tilde{p}: X \rightarrow \Phi(A)$  such that  $p \cong \tilde{p}(\phi_A, 1)$ .
4.  $\tilde{p}$  is the right lift  $p \blacktriangleleft \Phi(A)(\phi_A, 1)$ , and the canonical 2-cell  $\tilde{p}(\phi_A, 1) \Rightarrow p$  is an isomorphism.