The formal theories of presheaves and cocompletions

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Cocompletions

small category $\mathbf{A}$	+	small colimits	=	$\mathcal{P}[\mathbf{A}]$
FinSet	+	filtered colimits	=	**Set**
small finitely complete category	+	filtered colimits	=	locally finitely presentable category
small category	+	filtered colimits	=	finitely accessible category
FinSet	+	sifted colimits	=	**Set**
finitary algebraic theory $\mathcal{T}$	+	sifted colimits	=	$\mathcal{T}$-algebras
category $\mathbf{A}$	+	absolute colimits	=	Cauchy completion of $\mathbf{A}$

Cocompletions in **Cat**

Let Φ be a class of small categories.

- a category $\mathbf{E}$ is ***Φ-cocomplete*** if $\mathbf{E}$ has all colimits $colim_{i\in\mathbf{I}}\, f(i)$ with $\mathbf{I} \in \Phi$
- a functor $g\colon \mathbf{E} \to \mathbf{X}$ is ***Φ-cocontinuous*** if g preserves these colimits

Definition (Φ-cocompletion)

The ***Φ-cocompletion*** of a category $\mathbf{A}$ is a functor $\phi_{\mathbf{A}}\colon \mathbf{A} \to \Phi(\mathbf{A})$ such that:

1. $\Phi(\mathbf{A})$ is ***Φ-cocomplete***.
2. For all $f\colon \mathbf{A} \to \mathbf{X}$, with ***$\Phi$-cocomplete*** codomain, there is an essentially unique ***Φ-cocontinuous*** $\tilde{f}\colon \Phi(\mathbf{A}) \to \mathbf{X}$ such that $f \cong \tilde{f} \circ \phi_{\mathbf{A}}$.

Cocompletions in **Cat**

Example (Φ = filtered categories)

The ***filtered cocompletion*** of a category $\mathbf{A}$ is a functor $\phi_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{Ind}(\mathbf{A})$ such that:

1. $\mathrm{Ind}(\mathbf{A})$ has filtered colimits;
2. For all $f \colon \mathbf{A} \to \mathbf{X}$, such that $\mathbf{X}$ has filtered colimits, there is an essentially unique finitary $\tilde{f} \colon \mathrm{Ind}(\mathbf{A}) \to \mathbf{X}$ such that $f \cong \tilde{f} \circ \phi_{\mathbf{A}}$.

The filtered cocompletion of **FinSet** is

$$\mathbf{FinSet} \hookrightarrow \mathbf{Set}$$

There is plenty of work on cocompletions in $\mathbb{V}$-**Cat**, where $\mathbb{V}$ is a complete and cocomplete smcc.

NOTES ON ENRICHED CATEGORIES WITH COLIMITS OF SOME CLASS

G.M. KELLY AND V. SCHMITT

Virtual double categories

A *virtual double category* $\mathbb{X}$ has

1. *Objects* $X, A, \ldots$
2. Composable *tight-cells* $\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$
3. *Loose-cells* $p \colon Y \nrightarrow X$
4. Composable *2-cells*

$$\begin{tikzcd} X_0 \arrow[d, "f"'] & X_1 \arrow[l, "p_1"', "\shortmid" marking] & \cdots \arrow[l, "p_2"', "\shortmid" marking] & X_{n-1} \arrow[l, "p_{n-1}"', "\shortmid" marking] & X_n \arrow[l, "p_n"', "\shortmid" marking] \arrow[d, "g"] \\ A & & & & B \arrow[llll, "q", "\shortmid" marking] \arrow[llll, phantom, shift left=6, "\phi"] \end{tikzcd}$$

Example: $\mathbb{X} = \mathbb{V}$-**Cat**

1. $\mathbb{V}$-categories
2. $\mathbb{V}$-functors
3. $\mathbb{V}$-distributors, with $p(x, y) \in \mathbb{V}$
4. $\mathbb{V}$-natural transformations

$$p_1(x_0, x_1) \otimes \cdots \otimes p_n(x_{n-1}, x_n) \xrightarrow{\phi_{x_0, \ldots, x_n}} q(fx_0, gx_n)$$

Virtual equipments

A *virtual equipment* also has:

1. Loose-identities $E(1,1)$

$$\begin{tikzcd} & E \arrow[dl, equal] \arrow[dr, equal] & \\ E & \arrow[phantom, u, "\text{opcart}"] & E \arrow[ll, "E(1,1)", "\mid"'] \end{tikzcd}$$

2. Restrictions $p(f,g)$

$$\begin{tikzcd} X' \arrow[d, "f"'] \arrow[dr, phantom, "\text{cart}"] & Y' \arrow[l, "p(f,g)"'] \arrow[d, "g"] \\ X & Y \arrow[l, "p"] \end{tikzcd}$$

For $\mathbb{X} = \mathbb{V}$-**Cat**:

1. $E(1,1)(x,y) = E(x,y)$

2. $p(f,g)(x,y) = p(fx, gy)$

Restrictions of loose-identities $\quad E(f,g) = E(1,1)(f,g)$

$$\left.\begin{array}{rl} \text{Companions} & E(1_E, j) : A \nrightarrow E \\ \text{Conjoints} & E(j, 1_E) : E \nrightarrow A \end{array}\right\} j : A \to E$$

Strict restrictions

$$\begin{tikzcd} X' \arrow[d, "f"'] \arrow[dr, phantom, "\text{cart}"] & Y' \arrow[l, "p(f,g)"', "\shortmid"{marking}] \arrow[d, "g"] \\ X & Y \arrow[l, "p", "\shortmid"{marking}] \end{tikzcd}$$

We'll assume that restrictions are ***chosen*** and ***strict***:

$$p(1_X, 1_Y) = p \qquad p(f,g)(f',g') = p(ff', gg')$$

Globular 2-cells

Since we have restrictions, instead of 2-cells

$$\begin{tikzcd} X_0 \arrow[d, "f"'] & X_1 \arrow[l, "p_1"', "\shortmid" marking] & \cdots \arrow[l, "p_2"', "\shortmid" marking] \arrow[d, phantom, "\phi"] & X_{n-1} \arrow[l, "p_{n-1}"', "\shortmid" marking] & X_n \arrow[l, "p_n"', "\shortmid" marking] \arrow[d, "g"] \\ A & & & & B \arrow[llll, "q", "\shortmid" marking] \end{tikzcd}$$

we can consider *globular* 2-cells

$$\psi \colon p_1, \ldots, p_n \Rightarrow q(f, g)$$

$$\begin{tikzcd} X_0 \arrow[d, equal] & X_1 \arrow[l, "p_1"', "\shortmid" marking] & \cdots \arrow[l, "p_2"', "\shortmid" marking] \arrow[d, phantom, "\psi"] & X_{n-1} \arrow[l, "p_{n-1}"', "\shortmid" marking] & X_n \arrow[l, "p_n"', "\shortmid" marking] \arrow[d, equal] \\ X_0 & & & & X_n \arrow[llll, "q(f,g)", "\shortmid" marking] \end{tikzcd}$$

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Colimits in virtual equipments

Right lift $q \blacktriangleleft p$

$$\begin{tikzcd} X \arrow[dr, "p"', "|" marking] & Y \arrow[l, "q \blacktriangleleft p"', "|" marking] \arrow[d, "q", "|" marking] \arrow[dl, phantom, "\Rightarrow"] \\ & A \end{tikzcd}$$

such that

$$\frac{r_1, \ldots, r_n \Rightarrow q \blacktriangleleft p}{p, r_1, \ldots, r_n \Rightarrow q}$$

Weighted colimit $p \circledast f$

$$\begin{tikzcd} X \arrow[r, "p \circledast f"] \arrow[dr, "p"', "|" marking] & Y \\ & A \arrow[u, "f"'] \end{tikzcd}$$

such that $Y(p \circledast f, 1) \cong Y(f, 1) \blacktriangleleft p$

For $\mathbb{X} = \mathbb{V}$-**Cat**:

$$(q \blacktriangleleft p)(x, y) \cong \mathcal{P}A(p(-, x), q(-, y))$$

$$\cong \int_{a \in A} p(a, x) \multimap q(a, y)$$

assuming $\mathbb{V}$ is closed

$$Y((p \circledast f)x, y) \cong \mathcal{P}A(p(-, x), Y(f-, y))$$

Left extensions in virtual equipments

Weighted colimit $p \circledast f$

$$\begin{tikzcd} X \arrow[r, "p \circledast f"] \arrow[dr, "p"', "/" marking] & Y \\ & A \arrow[u, "f"'] \end{tikzcd}$$

such that $Y(p \circledast f, 1) \cong Y(f, 1) \blacktriangleleft p$

(Pointwise) left (Kan) extension $j \rhd f$

$$\begin{tikzcd} E \arrow[r, "j \rhd f"] & Y \\ & A \arrow[u, "f"'] \arrow[ul, "j"] \end{tikzcd}$$

such that $j \rhd f \cong E(j, 1) \circledast f$

For $\mathbb{X} = \mathbb{V}$-**Cat**:

$$Y((p \circledast f)x, y) \cong \mathcal{P}A(p(-, x), Y(f-, y))$$

$$Y((j \rhd f)x, y) \cong \mathcal{P}A(E(j-, x), Y(f-, y))$$

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2. For all $f \colon \mathbf{A} \to \mathbf{X}$, with ***$\Phi$-cocomplete*** codomain, there is an essentially unique ***Φ-cocontinuous*** $\tilde{f} \colon \Phi(\mathbf{A}) \to \mathbf{X}$ such that $f \cong \tilde{f} \circ \phi_{\mathbf{A}}$.

Cocompletions in a virtual equipment (wrong)

Let Φ be a class of loose-cells.

- an object E is ***Φ-cocomplete*** if E admits all colimits $p \circledast f$ with $p \in \Phi$
- a tight-cell $g \colon E \to X$ is ***Φ-cocontinuous*** if g preserves these colimits

Definition (Φ-cocompletion)

The ***Φ-cocompletion*** of an object A is a tight-cell $\phi_A \colon A \to \Phi(A)$ such that:

1. $\Phi(A)$ is ***Φ-cocomplete***.
2. For all $f \colon A \to X$, with ***Φ-cocomplete*** codomain, there is an essentially unique ***Φ-cocontinuous*** $\tilde{f} \colon \Phi(A) \to X$ such that $f \cong \tilde{f} \circ \phi_A$.

Cocompletions in a virtual equipment (still wrong)

A loose-cell $q \colon X \nrightarrow E$ is ***Φ-cocontinuous*** if $q(p \circledast f, 1) \cong q(f, 1) \blacktriangleleft p$ for $p \in \Phi$

Definition (Φ-cocompletion)

The ***Φ-cocompletion*** of A is a tight-cell $\phi_A \colon A \to \Phi(A)$ such that:

1. $\Phi(A)$ is ***Φ-cocomplete***.
2. For all $f \colon A \to X$, with ***Φ-cocomplete*** codomain, there is an essentially unique ***Φ-cocontinuous*** $\tilde{f} \colon \Phi(A) \to X$ such that $f \cong \tilde{f} \circ \phi_A$.
3. For all $p \colon X \nrightarrow A$, there is an essentially unique ***Φ-cocontinuous*** $\tilde{p} \colon X \nrightarrow \Phi(A)$ such that $p \cong \tilde{p}(\phi_A, 1)$.

Cocompletions in a virtual equipment

A loose-cell $q \colon X \nrightarrow E$ is ***Φ-cocontinuous*** if $q(p \circledast f, 1) \cong q(f, 1) \blacktriangleleft p$ for $p \in \Phi$

Definition (Φ-cocompletion)

The ***Φ-cocompletion*** of A is a tight-cell $\phi_A \colon A \to \Phi(A)$ such that:

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3. For all $p \colon X \nrightarrow A$, there is an essentially unique ***Φ-cocontinuous*** $\tilde{p} \colon X \nrightarrow \Phi(A)$ such that $p \cong \tilde{p}(\phi_A, 1)$.
4. $\tilde{p}$ is the right lift $p \blacktriangleleft \Phi(A)(\phi_A, 1)$, and the canonical 2-cell $\tilde{p}(\phi_A, 1) \Rightarrow p$ is an isomorphism.

ϕ_A is fully faithful

For every $p\colon X \nrightarrow A$,

$$\begin{aligned} p \blacktriangleleft \Phi(A)(\phi_A, \phi_A) &\cong (p \blacktriangleleft \Phi(A)(\phi_A, 1))(\phi_A, 1) \\ &\cong \tilde{p}(\phi_A, 1) \\ &\cong p \\ &\cong p \blacktriangleleft A(1,1) \end{aligned}$$

So there is a bijection $\dfrac{A(1,1) \Rightarrow p}{\Phi(A)(\phi_A, \phi_A) \Rightarrow p}$ natural in p.

Hence $A(1,1) \cong \Phi(A)(\phi_A, \phi_A)$.

Loose-composites in virtual equipments

Composite $q \odot p$

$$\begin{tikzcd} X & Y \arrow[l, "q"', "|" marking] \arrow[l, phantom, shift right=6, "\Downarrow"] & Z \arrow[l, "p"', "|" marking] \arrow[ll, bend left, "q \odot p", "|" marking] \end{tikzcd} \quad \text{such that} \quad \frac{r_1', \ldots, r_m', q \odot p, r_1, \ldots, r_n \Rightarrow s}{r_1', \ldots, r_m', q, p, r_1, \ldots, r_n \Rightarrow s}$$

Left-composite $q \odot_L p$

$$\begin{tikzcd} X & Y \arrow[l, "q"', "|" marking] \arrow[l, phantom, shift right=6, "\Downarrow"] & Z \arrow[l, "p"', "|" marking] \arrow[ll, bend left, "q \odot_L p", "|" marking] \end{tikzcd} \quad \text{such that} \quad \frac{q \odot_L p, r_1, \ldots, r_n \Rightarrow s}{q, p, r_1, \ldots, r_n \Rightarrow s}$$

Absolute colimits

For $j\colon A \to E$, a colimit $q \circledast f$ in E is *j-absolute* if

$$E(j, q \circledast f) \cong E(j, f) \odot_L q$$

If $q \circledast f$ is j-absolute, then $q \circledast f$ is preserved by every $j \rhd g$

Absolute colimits

For $j\colon A \to E$, a colimit $q \circledast f$ in E is *j-absolute* if

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Φ-colimits in $\Phi(A)$ are ϕ_A-absolute:

$$\frac{\frac{\frac{\frac{\frac{\Phi(A)(\phi_A, q \circledast f), r_1, \ldots, r_n \Rightarrow s}{r_1, \ldots, r_n \Rightarrow s \blacktriangleleft \Phi(A)(\phi_A, q \circledast f)}}{r_1, \ldots, r_n \Rightarrow (s \blacktriangleleft \Phi(A)(\phi_A, 1))(q \circledast f, 1)}}{r_1, \ldots, r_n \Rightarrow (s \blacktriangleleft \Phi(A)(\phi_A, f)) \blacktriangleleft q}}{q, r_1, \ldots, r_n \Rightarrow s \blacktriangleleft \Phi(A)(\phi_A, f)}}{\Phi(A)(\phi_A, f), q, r_1, \ldots, r_n \Rightarrow s}$$

Φ-cocontinous tight-cells

The following are equivalent for a tight-cell $g \colon \Phi(A) \to X$:

1. g is Φ-cocontinuous;
2. g is a left extension $\phi_A \rhd f$, for some $f \colon A \to X$;
3. g preserves ϕ_A-absolute colimits.

Presheaves in virtual equipments

Definition (Presheaf object)

The presheaf object for A is a loose-cell $\pi_A \colon \mathcal{P}[A] \nrightarrow A$ such that:

1. For all $q \colon Y \nrightarrow A$, there is a unique tight-cell $\breve{q} : Y \to \mathcal{P}[A]$ such that $\pi_A(1, \breve{q}) = q$.
2. π_A is dense ($\pi_A \blacktriangleleft \pi_A \cong \mathcal{P}[A](1,1)$).

For $\mathbb{X} = \mathbb{V}$-**Cat**:

- Objects of $\mathcal{P}[A]$ are $\mathbb{V}$-presheaves ϕ on A
- $\mathcal{P}[A](\phi, \psi) \cong \int_{a \in A} \phi(a) \multimap \psi(a)$

  assuming $\mathbb{V}$ is closed
- $\pi_A(a, \phi) = \phi(a)$

Presheaves in virtual equipments

Definition (Presheaf object)

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  assuming $\mathbb{V}$ is closed
- $\pi_A(a, \phi) = \phi(a)$

- Right lifts: $q \blacktriangleleft p \cong \mathcal{P}[A](\breve{p}, \breve{q})$

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For $\mathbb{X} = \mathbb{V}$-**Cat**:

- Objects of $\mathcal{P}[A]$ are $\mathbb{V}$-presheaves ϕ on A
- $\mathcal{P}[A](\phi, \psi) \cong \int_{a \in A} \phi(a) \multimap \psi(a)$

  assuming $\mathbb{V}$ is closed
- $\pi_A(a, \phi) = \phi(a)$

- Right lifts: $q \blacktriangleleft p \cong \mathcal{P}[A](\breve{p}, \breve{q})$
- Yoneda lemma: $\mathcal{P}[A](\text{よ}_A, \breve{q}) \cong q$, where $\text{よ}_A = \breve{A(1,1)} : A \to \mathcal{P}[A]$

Presheaves in virtual equipments

Definition (Φ-presheaf object)

The Φ-presheaf object for A is a loose-cell $\pi_A \colon \Phi[A] \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} A$ such that:

1. For all $q \colon Y \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} A$ with $q \in \Phi$, there is a unique tight-cell $\breve{q} : Y \to \Phi[A]$ such that $\pi_A(1, \breve{q}) = q$.
2. π_A is dense ($\pi_A \blacktriangleleft \pi_A \cong \Phi[A](1,1)$).
3. $\pi_A(1, f) \in \Phi$, for all $f \colon Y \to \Phi[A]$.

- Right lifts: $q \blacktriangleleft p \cong \Phi[A](\breve{p}, \breve{q})$, assuming $p, q \in \Phi$
- Yoneda lemma: $\Phi[A](\phi_A, \breve{q}) \cong q$, where $\phi_A = A(1,1) \colon A \to \Phi[A]$, assuming $A(1,1) \in \Phi$ and $q \in \Phi$

Cocompletions are almost presheaf objects

Let $\phi_A \colon A \to \Phi(A)$ be a Φ-cocompletion, and assume every $\tilde{p} \cong p \blacktriangleleft \Phi(A)(\phi_A, 1)$ is representable:

$$\tilde{p} \cong \Phi(A)(1, \breve{p})$$

Then $\breve{p}$ is essentially unique such that $\pi_A(1, \breve{p}) \cong p$, where $\pi_A \cong \Phi(A)(\phi_A, 1)$

$$\begin{array}{rcl} p \colon X \nrightarrow A & \mapsto & \breve{p} \colon X \to \Phi(A) \\ f \colon X \to \Phi(A) & \mapsto & \Phi(A)(\phi_A, f) \colon X \nrightarrow A \end{array}$$

Colimits in presheaf objects

The colimit

$$
\begin{tikzcd}
X \arrow[r, "q \circledast f"] \arrow[dr, "q"', "|" marking] & \Phi[A] \\
& A \arrow[u, "f"']
\end{tikzcd}
$$

is given by a left-composite

$$q \circledast f \cong \wideparen{\pi_A(1, f) \odot_L q}$$

assuming $\pi_A(1, f) \odot_L q$ exists and is in Φ.

This is ϕ_A-absolute:

$$\Phi[A](\phi_A, q \circledast f) \cong \Phi[A](\phi_A, f) \odot_L q$$

Cocompletions from presheaves

Assuming:

1. The presheaf object $\pi_A \colon \Phi[A] \underbrace{\nrightarrow A}$ exists.
2. $A(1,1) \in \Phi$ (so we have $\phi_A = A(1,1) \colon A \to \Phi[A]$).
3. Φ is closed under left-composites (so $\Phi[A]$ has Φ-colimits).
4. $q \blacktriangleleft p$ exists for all $p \in \Phi$ (in particular, for $p \cong \Phi[A](\phi_A, 1)$).

The Φ-cocompletion of A is $\phi_A \colon A \to \Phi[A]$.

Cocompletions from presheaves

Given Ψ, if there exists some Φ such that:

1. The presheaf object $\pi_A \colon \Phi[A] \nrightarrow A$ exists.
2. $A(1,1) \in \Phi$ (so we have $\phi_A = A(1,1) \colon A \to \Phi[A]$).
3. Φ is closed under left-composites (so $\Phi[A]$ has Φ-colimits).
4. $q \blacktriangleleft p$ exists for all $p \in \Phi$ (in particular, for $p \cong \Phi[A](\phi_A, 1)$).
5. Ψ and Φ are colimit-equivalent.

The Ψ-cocompletion of A is $\phi_A \colon A \to \Phi[A]$.

Cocompletions from presheaves, in $\mathbb{V}$-**Cat**

If A is small, each $p \in \Phi$ has small codomain, and the $\mathbb{V}$-category $\mathcal{P}[A]$ exists:

$\Phi(A) \hookrightarrow \mathcal{P}[A]$ is the smallest full sub-$\mathbb{V}$-category such that

- $\Phi(A)$ contains the representables
- $\Phi(A)$ is closed under Φ-colimits

$\phi_A : A \to \Phi(A)$ is the Yoneda embedding

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