Flexible presentations of graded monads

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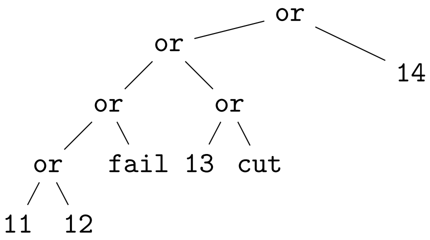
Example: nondeterminism with backtracking and cut

$$\texttt{or}(\texttt{or}(\texttt{or}(\texttt{or}(\texttt{return11}, \texttt{return12}), \texttt{fail}), \texttt{or}(\texttt{return13}, \texttt{cut})), \texttt{return14})$$



These computations satisfy some equations:

$$\texttt{or}(x,y) \equiv x \qquad \text{whenever } x \text{ definitely cuts}$$

Models of effects from presentations

1. Effects can be modelled using monads [Moggi '89]
2. which often come from presentations [Plotkin and Power '02]
3. which induce algebraic operations [Plotkin and Power '03]

Example: (based on [Piróg and Staton '17])

1. Nondeterminism with can be modelled using a monad Cut
$$\mathrm{Cut}X = \mathrm{List}X \times \{\mathsf{cut}, \mathsf{nocut}\}$$
2. which comes from the presentation of monoids with a left zero:
$$\mathsf{or} : 2 \qquad \mathsf{fail} : 0 \qquad \mathsf{cut} : 0$$
$$\mathsf{or}(\mathsf{or}(x,y),z) = \mathsf{or}(x,\mathsf{or}(y,z)) \qquad \mathsf{or}(\mathsf{fail},x) = x = \mathsf{or}(x,\mathsf{fail}) \qquad \mathsf{or}(\mathsf{cut},x) = x$$
3. which induces algebraic operations
$$\mathrm{or}_X \ : \ \mathrm{Cut}\,X \times \mathrm{Cut}\,X \to \mathrm{Cut}\,X$$
$$\mathrm{fail}_X \ : \ 1 \to \mathrm{Cut}\,X \qquad \mathrm{cut}_X \ : \ 1 \to \mathrm{Cut}\,X$$

Example: grading nondeterminism with backtracking and cut

$$\texttt{or}(x,y) \equiv y \qquad \text{whenever } x \text{ has grade } \bot$$

Assign grades $e \in \{\bot, 1, \top\}$ to computations:

$\top$ don't know anything

$\vee|$

1 definitely cuts or `returns` something

$\vee|$

$\bot$ definitely cuts

Graded monad Cut:

$$\mathrm{Cut}Xe = \{(\mathrm{xs}, c) \in \mathrm{List}X \times \{\mathsf{cut}, \mathsf{nocut}\} \mid (e = \bot \Rightarrow c = \mathsf{cut}) \wedge (e = 1 \Rightarrow c = \mathsf{cut} \vee \mathrm{xs} \neq [])\}$$

Kleisli extension:

$$\frac{f : X \to \mathrm{Cut}Ye}{f^\dagger_d : \mathrm{Cut}Xd \to \mathrm{Cut}Y(d \cdot e)} \quad \text{where} \quad \begin{array}{rcl} \top \cdot e & = & \top \\ 1 \cdot e & = & e \\ \bot \cdot e & = & \bot \end{array}$$

Example: grading nondeterminism with backtracking and cut

1. Nondeterminism with cut can be modelled using a graded monad Cut

$$\begin{aligned}\mathrm{Cut}\,X\,e = \{(\mathrm{xs}, c) \in \mathrm{List}\,X \times \{\mathsf{cut}, \mathsf{nocut}\} \\ \mid (e = \bot \Rightarrow c = \mathsf{cut}) \\ \wedge\ (e = 1 \Rightarrow c = \mathsf{cut} \vee \mathrm{xs} \neq [\,])\}\end{aligned}$$

2. which comes from a graded presentation of monoids with a left zero?
3. which induces graded algebraic operations?

$$\mathrm{or}_{d_1,d_2,X} \;:\; \mathrm{Cut}\,X\,d_1 \times \mathrm{Cut}\,X\,d_2 \to \mathrm{Cut}\,X\,(d_1 \sqcap d_2) \qquad (d_1, d_2 \in \{\bot, 1, \top\})$$

$$\mathrm{fail}_X \;:\; 1 \to \mathrm{Cut}\,X\,\top \qquad \mathrm{cut}_X \;:\; 1 \to \mathrm{Cut}\,X\,\bot$$

The existing notions of graded presentation are not general enough

[Smirnov '08, Milius et al. '15, Dorsch et al. '19, Kura '20]

This work

Develop a notion of flexibly graded presentation

- Every flexibly graded presentation (Σ, E) induces
  - a canonical graded monad $\mathsf{T}_{(\Sigma,E)}$
  - along with a flexibly graded algebraic operation for each operation of the presentation
- Examples like Cut have computationally natural flexibly graded presentations
- The constructions are mathematically justified by locally graded categories, and a notion of flexibly graded abstract clone

Flexibly graded presentations

Given an ordered monoid $(\mathbb{E}, \leq, 1, \cdot)$ of grades,

a flexibly graded presentation (Σ, E) consists of

- a signature Σ: sets

  $$\Sigma(d'_1, \ldots, d'_n; d)$$

  of operations

  $$\frac{e \in \mathbb{E} \quad \Gamma \vdash t_1 : d'_1 \cdot e \quad \cdots \quad \Gamma \vdash t_n : d'_n \cdot e}{\Gamma \vdash \mathrm{op}(e; t_1, \ldots, t_n) : d \cdot e}$$

- a collection of axioms E: sets

  $$E(d'_1, \ldots, d'_n; d)$$

  of equations

  $$x_1 : d'_1, \ldots, x_n : d'_n \vdash t \equiv u : d$$

Part of the presentation of nondeterminism with cut:

$$\text{grades } \mathbb{E} = \{\bot \leq 1 \leq \top\}$$

$$\frac{\Gamma \vdash t_1 : d'_1 \cdot e \qquad \Gamma \vdash t_2 : d'_2 \cdot e}{\Gamma \vdash \mathrm{or}_{d'_1, d'_2}(e; t_1, t_2) : (d'_1 \sqcap d'_2) \cdot e}$$

$$\mathrm{or}_{\bot, e}(1; x, y) \equiv x$$

Semantics

For every flexibly graded presentation (Σ, E), there is:

- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$ ← [Wood '76]
- a sound and complete equational logic
- a graded monad $\mathsf{T}_{(\Sigma,E)}$ on $\mathbf{Set}$ and concrete functor $R_{(\Sigma,E)} : \mathbf{Alg}(\Sigma, E) \to \mathbf{EM}(\mathsf{T}_{(\Sigma,E)})$, satisfying a universal property

$$\begin{array}{ccc} \mathbf{Alg}(\Sigma, E) & \xrightarrow{R_{(\Sigma,E)}} & \mathbf{EM}(\mathsf{T}_{(\Sigma,E)}) \\ & {\scriptstyle R'} \searrow & \downarrow {\scriptstyle \mathbf{EM}(\alpha)} \\ & & \mathbf{EM}(\mathsf{T}') \end{array} \qquad \begin{array}{c} \mathsf{T}_{(\Sigma,E)} \\ \uparrow {\scriptstyle \alpha} \\ \mathsf{T}' \end{array}$$

- for every op in Σ, a flexibly graded algebraic operation for $\mathsf{T}_{(\Sigma,E)}$

A large class of graded monads have flexibly graded presentations:

- exactly the graded monads on $\mathbf{Set}$ that preserve conical sifted colimits