Flexible presentations of graded monads

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Presentations of monads

Presentation:

operations $\mathrm{op} : n$

$+$ equations $t \equiv u$

Presentation of monoids:

$$m : 2 \quad u : 0$$

$$m(u(), x) \equiv x \equiv m(x, u())$$

$$m(m(x, y), z) \equiv m(x, m(y, z))$$

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Algebra:

set A with
functions $[\![\mathrm{op}]\!] : A^n \longrightarrow A$
satisfying equations

Monoid:

set A with
functions $[\![u]\!] : 1 \longrightarrow A$, $[\![m]\!] : A \times A \longrightarrow A$
satisfying unit and associativity eqns

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Free algebra on X:

algebra $(TX, [\![-]\!])$ with
function $\eta_X : X \to TX$
satisfying universal property

Free monoid on X:

monoid $(\mathrm{List}\, X, [], +\!\!+)$ with
singleton function $X \to \mathrm{List}\, X$

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Free algebra monad T:
has the same algebras as the presentation

Free monoid monad List:

has monoids as algebras

Grading

Definition

A *graded set* $X : \mathbb{N}_{\leq} \longrightarrow \mathbf{Set}$ consists of:

- a set Xe for each $e \in \mathbb{N}$
- a function $X(e{\leq}e') : Xe \to Xe'$ for each $e \leq e' \in \mathbb{N}$

such that $X(e \leq e) = \mathrm{id}$ and $X(e'{\leq}e'') \circ X(e{\leq}e') = X(e{\leq}e'')$.

Example

- $\mathrm{List}Xe$ is lists over X of length $\leq e$
- $\mathrm{List}X(e{\leq}e')$ is the inclusion $\mathrm{List}Xe \subseteq \mathrm{List}Xe'$

Grading

Definition

A *graded monoid* (A, m, u) consists of:

- a graded set $A : \mathbb{N}_{\leq} \longrightarrow \mathbf{Set}$
- multiplication functions $m_{e_1,e_2} : Ae_1 \times Ae_2 \longrightarrow A(e_1 + e_2)$ natural in $e_1, e_2 \in \mathbb{N}_{\leq}$
- a unit $u \in A0$

such that

$$m_{0,e}(u, x) = x = m_{e,0}(x, u)$$

$$m_{e_1+e_2,e_3}(m_{e_1,e_2}(x, y), z) = m_{e_1,e_2+e_3}(x, m_{e_2,e_3}(y, z))$$

Example

- graded set $\mathrm{List}X$
- multiplication $(+\!\!+) : \mathrm{List}Xe_1 \times \mathrm{List}Xe_2 \longrightarrow \mathrm{List}X(e_1 + e_2)$
- unit $[] \in \mathrm{List}X0$

Grading

Definition (Smirnov '08, Melliès '12, Katsumata '14)

A *graded monad* T consists of:

- a graded set TX for each (ungraded) set X
- unit functions $\eta_X : X \to TX1$
- Kleisli extension $\dfrac{f : X \to TYe}{f_d^\dagger : TXd \to TY(d \cdot e)}$ natural in d, e

such that the monad laws hold:

$$f_1^\dagger \circ \eta_X = f \qquad (\eta_X)_d^\dagger = \mathrm{id}_{TXm} \qquad (g_e^\dagger \circ f)_d^\dagger = g_{d \cdot e}^\dagger \circ f_d^\dagger$$

Example

- $\mathrm{List}X$ for each set X
- singleton functions $X \to \mathrm{List}X1$
- $f_d^\dagger[x_1, \ldots, x_k] = f x_1 +\!\!+ \cdots +\!\!+ f x_k$

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Example

- $\mathrm{List}X$ for each set X
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This is the free graded monoid graded monad

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This is the free graded monoid graded monad

- but its algebras are not graded monoids in general

(Rigidly) graded presentations [Smirnov '08, Dorsch et al. 19, Kura '20]

- Each operation op has an arity $n \in \mathbb{N}$ and grade $e' \in \mathbb{N}$
- Terms generated by variables, coercions, and

$$\frac{\Gamma \vdash t_1 : e \qquad \cdots \qquad \Gamma \vdash t_n : e}{\Gamma \vdash \mathrm{op}(t_1, \ldots, t_n) : e' \cdot e}$$

- There is an algebra-preserving correspondence between graded presentations and a class of graded monads

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But we want something like

$$\frac{\Gamma \vdash t_1 : e_1 \qquad \Gamma \vdash t_2 : e_2}{\Gamma \vdash m(t_1, t_2) : e_1 + e_2}$$

Flexibly graded presentations

- Each operation op has a list of grades $e_1, \ldots, e_n$, and another grade e'
- Terms generated by variables, coercions, and

$$\frac{\Gamma \vdash t_1 : e_1 \qquad \cdots \qquad \Gamma \vdash t_n : e_n}{\Gamma \vdash \mathrm{op}(t_1, \ldots, t_n) : e'}$$

Flexibly graded presentations

Definition

A *flexibly graded signature* consists of a graded set $\Sigma_{\vec{e}}$ for each $\vec{e}$.

Given a signature Σ, terms in context $\Gamma = x_1 : e_1, \ldots, x_n : e_n$ are generated by

$$\frac{}{\Gamma \vdash [e_n \leq e'] x_i : e'} \ (e_i \leq e' \in \mathbb{N})$$

$$\frac{\Gamma \vdash t_1 : e_1 \qquad \cdots \qquad \Gamma \vdash t_n : e_n}{\Gamma \vdash \mathrm{op}(t_1, \ldots, t_n) : e'} \ (\mathrm{op} \in \Sigma_{\vec{e}}\, e')$$

Then $\mathrm{Tm}^{\Sigma}_{\vec{e}}$ is a graded set, where

$$\mathrm{Tm}^{\Sigma}_{\vec{e}}\, e' = \text{set of terms } x_1 : e_1, \ldots, x_n : e_n \vdash t : e'$$

Flexibly graded presentations

Definition

A *flexibly graded presentation* consists of

1. a flexibly graded signature Σ
2. sets of axioms (pairs of terms $t, u \in \mathrm{Tm}^{\Sigma}_{\vec{e}} e'$)
3. ...

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The Kleisli extension

$$\frac{f : X \to \mathrm{List} Y e}{f^{\dagger}_{d} : \mathrm{List} X d \to \mathrm{List} Y (d \cdot e)}$$

satisfies

$$f^{\dagger}_{d}(\mathrm{xs} +\!\!+_{e_1, e_2} \mathrm{ys}) = f^{\dagger}_{d} \mathrm{xs} +\!\!+_{e_1 \cdot d, e_2 \cdot d} f^{\dagger}_{d} \mathrm{ys}$$

Flexibly graded presentations

Definition

A *flexibly graded presentation* consists of

1. a flexibly graded signature Σ
2. sets of axioms (pairs of terms $t, u \in \mathrm{Tm}^{\Sigma}_{\vec{e}} e'$)
3. for each $\mathrm{op} \in \Sigma_{\vec{e}} e'$ and $d \in \mathbb{N}$, a term $\langle\!\langle \mathrm{op}, d \rangle\!\rangle_{e'} \in \mathrm{Tm}^{\Sigma}_{\vec{e} \cdot d}(e' \cdot d)$, natural in d

such that

4. $\langle\!\langle \mathrm{op}, - \rangle\!\rangle$ respects 1, $\cdot$ and $\leq$
5. $\langle\!\langle t, d \rangle\!\rangle \equiv \langle\!\langle u, d \rangle\!\rangle$ is admissible for every axiom $t \equiv u$ and d (using $\langle\!\langle -, - \rangle\!\rangle$ lifted to terms)

Presentation of graded monoids

1. Signature: $u \in \Sigma_{()} e'$ for each e', and $m_{e_1,e_2} \in \Sigma_{(e_1,e_2)} e'$ for each $e' \geq e_1 + e_2$
2. Axioms:

$$m_{e_1',e_2'}([e_1 \leq e_1']x_1, [e_2 \leq e_2']x_2) \equiv [(e_1 \cdot e_2) \leq (e_1' \cdot e_2')](m_{e_1,e_2}(x_1, x_2))$$

$$m_{0,e}(u(), x) \equiv x \qquad x \equiv m_{e,0}(x, u())$$

$$m_{e_1+e_2,e_3}(m_{e_1,e_2}(x_1, x_2), x_3) \equiv m_{e_1,e_2+e_3}(x_1, m_{e_2,e_3}(x_2, x_3))$$

3. $\langle\!\langle u, d \rangle\!\rangle = u$ and $\langle\!\langle m_{e_1,e_2}, d \rangle\!\rangle = m_{e_1 \cdot d, e_2 \cdot d}(x_1, x_2)$

Algebras

Definition

Given a flexibly graded presentation, an *algebra* consists of

- a graded set A
- a function $[\![\mathrm{op}]\!]_{e'} : \prod_i Ae_i \to Ae'$ for each $\mathrm{op} \in \Sigma_{\vec{e}}e'$, natural in e'

such that $[\![t]\!] = [\![u]\!]$ for every axiom $t \equiv u$.

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such that $[\![t]\!] = [\![u]\!]$ for every axiom $t \equiv u$.

Theorem

- *For every presentation, there is a graded monad with the closest algebras possible*
- *For every sifted-cocontinuous graded monad, there is a presentation with the same algebras*

Locally graded categories [Wood '76]

Definition

A *locally graded category* $\mathcal{C}$ consists of

- a collection $|\mathcal{C}|$ of objects
- graded sets $\mathcal{C}(X, Y)$ of morphisms
  ($f : X -e\to Y$ means $f \in \mathcal{C}(X, Y)e$)
- identities $\mathrm{id}_X : X -1\to X$
- composition

$$\frac{f : X -e\to Y \qquad g : Y -e'\to Z}{g \circ f : X -e \cdot e'\to Z}$$

  natural in e, e'

such that

$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

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(These are categories enriched over $[\mathbb{N}_{\leq}, \mathbf{Set}]$ with Day convolution)

Locally graded categories

The locally graded category $\mathbf{GObj}(\mathbf{Set})$:

- Objects are graded sets
- Morphisms $f : X \overset{e}{\rightarrow} Y$ are families of functions $f_d : Xd \to Y(d \cdot e)$, natural in d
- Identities are the identity functions
- Composition $g \circ f$ is

$$(g \circ f)_d : Xd \xrightarrow{f_d} Y(d \cdot e) \xrightarrow{g_{d \cdot e}} Z(d \cdot e \cdot e')$$

For example, $(\lambda x.\, [x, x, x])^{\dagger} : \mathrm{List}X \overset{3}{\rightarrow} \mathrm{List}X$

$$[x_1, \ldots, x_n] \mapsto [x_1, x_1, x_1, \ldots, x_n, x_n, x_n]$$

Locally graded categories

The locally graded category $\mathbf{Free}(\mathbf{Set})$:

- Objects are sets
- Morphisms are given by

$$\mathbf{Free}(\mathbf{Set})(X, Y)d = \begin{cases} \mathbf{Set}(X, Y) & \text{if } d \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Identities and composition are as in $\mathbf{Set}$

Functors

Definition

A *functor* $F : \mathcal{C} \to \mathcal{D}$ between locally graded categories is an object mapping $F : |\mathcal{C}| \to |\mathcal{D}|$ with a mapping of morphisms

$$\frac{f : X \xrightarrow{e} Y}{Ff : FX \xrightarrow{e} FY}$$

natural in e, and preserving identities and composition.

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Functor $K : \mathbf{Free}(\mathbf{Set}) \longrightarrow \mathbf{GObj}(\mathbf{Set})$:

$$KXd = \begin{cases} X & \text{if } d \geq 1 \\ 0 & \text{otherwise} \end{cases} \qquad (Kf)_d = \begin{cases} f & \text{if } d \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

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$$\frac{KX \overset{e}{\rightarrow} Y}{\overline{X \longrightarrow Ye}}$$

Relative monads [Altenkirch, Chapman, Uustalu '15]

Definition

A *J-relative monad* T (for $J : \mathcal{J} \to \mathcal{C}$) consists of:

- object mapping $T : |\mathcal{J}| \to |\mathcal{C}|$
- unit morphisms $\eta_X : JX \multimap_1 TX$
- Kleisli extension $\dfrac{f : JX \multimap_e TY}{f^\dagger : TX \multimap_e TY}$ natural in e

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Each has an Eilenberg-Moore construction

$$U_\mathsf{T} : \mathbf{EM}(\mathsf{T}) \to \mathbf{GObj}(\mathbf{Set})$$

satisfying nice properties

$(K : \mathbf{Free}(\mathbf{Set}) \to \mathbf{GObj}(\mathbf{Set}))$-relative monads

These are just graded monads:

- Assignment on objects:

$$T : |\mathbf{Free}(\mathbf{Set})| \to |\mathbf{GObj}(\mathbf{Set})|$$

- Unit:

$$\eta_X : KX \xrightarrow{1} TX$$

- Kleisli extension:

$$\frac{f : KX \xrightarrow{e} TY}{f^\dagger : TX \xrightarrow{e} TY}$$

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A *flexibly graded monad* is a monad on $\mathbf{GObj}(\mathbf{Set})$, i.e. a $(\mathrm{Id}_{\mathbf{GObj}(\mathbf{Set})} : \mathbf{GObj}(\mathbf{Set}) \to \mathbf{GObj}(\mathbf{Set}))$-relative monad.

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Example

$\mathrm{List}_{\mathrm{flex}}$ has graded monoids as algebras

$$\mathrm{List}_{\mathrm{flex}} X e = \mathrm{colim}_{\vec{n} \in S_e} \prod_i X n_i$$

where S_e is lists $(n_1, \ldots, n_k)$ with sum $\leq e$.

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Theorem

There is an algebra-preserving correspondence between

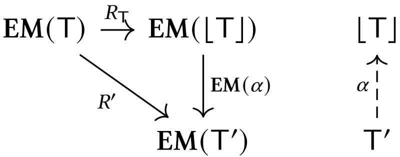
- *flexibly graded presentations*
- *flexibly graded monads that preserve conical sifted colimits*

Flexibly graded to rigidly graded

Every flexibly graded monad T restricts to a (rigidly) graded monad ⌊T⌋ by

$$\lfloor T \rfloor X = T(KX)$$

This is universal:



(where R_T, R' are over $\mathbf{GObj}(\mathbf{Set})$)

and free ⌊T⌋-algebras are free T-algebras

Example: $\lfloor \mathrm{List}_{\mathrm{flex}} \rfloor \cong \mathrm{List}$

Presenting graded monads

Given a flexibly graded presentation:

1. there is a flexibly graded monad T with the same algebras
2. so there is a universal graded monad $\lfloor \mathsf{T} \rfloor$
3. and free $\lfloor \mathsf{T} \rfloor$-algebras form free algebras for the presentation

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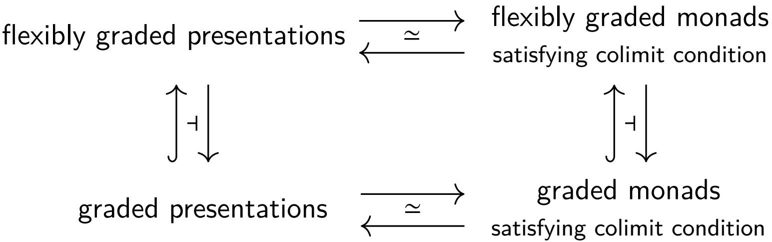
For the presentation of graded monoids:

1. the flexibly graded monad is $\mathsf{List}_{\mathrm{flex}}$
2. the universal graded monad is $\lfloor \mathsf{List}_{\mathrm{flex}} \rfloor \cong \mathsf{List}$
3. so the free List-algebras $\mathrm{List}\,X$ form free graded monoids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