Flexible presentations of graded monads

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Joint work with Shin-ya Katsumata, Tarmo Uustalu and Nicolas Wu

Motivation

1. Effects can be modelled using monads

   [Moggi '89]
2. which often come from presentations [Plotkin and Power '02]
3. which induce algebraic operations [Plotkin and Power '03]

Example:

1. Nondeterministic computations can be modelled using the free monoid monad List
2. which comes from the presentation of monoids

$$\mathsf{fail} : 0 \qquad \mathsf{or} : 2$$

$$\mathsf{or}(\mathsf{fail}, x) = x = \mathsf{or}(x, \mathsf{fail}) \qquad \mathsf{or}(\mathsf{or}(x, y), z) = \mathsf{or}(x, \mathsf{or}(y, z))$$

3. which induces algebraic operations

$$\mathrm{fail}_X = (\lambda\_.\, [\,]) \;:\; 1 \longrightarrow \mathrm{List}\, X$$

$$\mathrm{or}_X = (\lambda(\mathrm{xs}, \mathrm{ys}).\, \mathrm{xs} +\!\!+\, \mathrm{ys}) \;:\; \mathrm{List}\, X \times \mathrm{List}\, X \longrightarrow \mathrm{List}\, X$$

Motivation

1. Effects with quantitative information can be modelled using graded monads
   [Katsumata '14]
2. which often come from graded presentations?
   [Smirnov '08, Milius et al. '15, Dorsch et al. '19, Kura '20]
3. which induce algebraic operations?

Goal

Develop a notion of flexibly graded presentation for graded monads

Each flexibly graded presentation (Σ, E) induces

1. a flexibly graded (abstract) clone of terms
2. hence an $[\mathbb{E}, \mathbf{Set}]$-monad on $\mathbf{GSet}$
3. hence a graded monad

Graded monoids

A graded monoid A is

- a functor $A : \mathbb{N}_{\leq} \longrightarrow \mathbf{Set}$
- with an element $u \in A0$
- and a natural transformation

$$m_{d_1,d_2} : Ad_1 \times Ad_2 \longrightarrow A(d_1 + d_2)$$

such that

$$m_{0,d}(u, x) = x = m_{d,0}(x, u) \qquad m_{d_1+d_2,d_3}(m_{d_1,d_2}(x, y), z) = m_{d_1,d_2+d_3}(x, m_{d_2,d_3}(y, z))$$

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A *morphism* f : A $\longrightarrow$ A$'$ of grade $e \in \mathbb{N}$ is a natural transformation

$$f : A \Rightarrow A'(- \cdot e)$$

preserving the structure:

$$f_0(u) = u' \qquad f_{d_1+d_2}(m_{d_1,d_2}(x, y)) = m'_{d_1,d_2}(f_{d_1}(x), f_{d_2}(y))$$

So we get a $[\mathbb{N}_{\leq}, \mathbf{Set}]$-category $\mathbf{GMon}$, and $U : \mathbf{GMon} \longrightarrow \mathbf{GSet}$

Grading via $[\mathbb{E}, \mathbf{Set}]$-categories

Let $(\mathbb{E}, 1, \cdot)$ be a small strict monoidal category of grades

- for example $\mathbb{N}_{\leq}$ with multiplication

- $[\mathbb{E}, \mathbf{Set}]$ is a monoidal category with Day convolution
- we work in $[\mathbb{E}, \mathbf{Set}]$-$\mathbf{CAT}$

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The $[\mathbb{E}, \mathbf{Set}]$-category $\mathbf{GSet}$ is $[\mathbb{E}, \mathbf{Set}]$ enriched over itself:

- objects are graded sets $X : \mathbb{E} \longrightarrow \mathbf{Set}$
- morphisms of grade e (elements of $\mathbf{GSet}(X, Y)e$) are natural transformations

$$f : X \Rightarrow Y(- \cdot e)$$

- identities $\mathrm{id}_X \in \mathbf{GSet}(X, X)1$
- composition

$$(g \circ f) : X \xRightarrow{f} Y(- \cdot e) \xRightarrow{g_{- \cdot e}} Z(- \cdot e \cdot e')$$

where $f \in \mathbf{GSet}(X, Y)e$ and $g \in \mathbf{GSet}(Y, Z)e'$

Relative monads in $[\mathbb{E}, \mathbf{Set}]$-$\mathbf{CAT}$

A $(J : \mathcal{A} \to C)$-relative monad T is: [Altenkirch, Chapman, Uustalu '15]

- a function $T : |\mathcal{A}| \to |C|$
- with a morphism $\eta_X : JX \to TX$ for each $X \in |\mathcal{A}|$
- and a natural transformation

$$(-)^\dagger : C(JX, TY) \Rightarrow C(TX, TY)$$

  for each $X, Y \in |\mathcal{A}|$

such that

$$f^\dagger \circ \eta_X = f \qquad {\eta_X}^\dagger = \mathrm{id}_X \qquad (g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger$$

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T has an Eilenberg-Moore $[\mathbb{E}, \mathbf{Set}]$-category, and a forgetful $[\mathbb{E}, \mathbf{Set}]$-functor

$$U_{\mathsf{T}} : \mathbf{EM}(\mathsf{T}) \to \mathcal{C}$$

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We rely heavily on some general results about relative monads (jww Nathanael Arkor)

Graded monads

For each set X, define

$$JX = \mathbb{E}(1, -) \bullet X \quad : \quad \mathbb{E} \to \mathbf{Set}$$

so that

$$\mathbf{GSet}(JX, A)e \cong \mathbf{Set}(X, Ae)$$

and form a fully faithful $[\mathbb{E}, \mathbf{Set}]$-functor

$$J : \mathbf{RSet} \to \mathbf{GSet}$$

($\mathbf{RSet}$ is the free $[\mathbb{E}, \mathbf{Set}]$-category on $\mathbf{Set}$)

Definition: an $\mathbb{E}$-graded monad (on $\mathbf{Set}$) is a J-relative monad
(This is equivalent to the definitions in [Smirnov '08, Melliès '12, Katsumata '14])

Flexibly graded clones

For each finite sequence $d_1, \ldots, d_n \in |\mathbb{E}|$, define

$$K(d_1, \ldots, d_n) = \coprod_i \mathbb{E}(d_i, -) \quad : \quad \mathbb{E} \longrightarrow \mathbf{Set}$$

so that

$$\mathbf{GSet}(K(d_1, \ldots, d_n), A)e \cong \prod_i A(d_i \cdot e)$$

and form a fully faithful $[\mathbb{E}, \mathbf{Set}]$-functor

$$K : \mathbf{FCtx} \longrightarrow \mathbf{GSet}$$

is a fully faithful $[\mathbb{E}, \mathbf{Set}]$-functor

A flexibly graded (abstract) clone is a K-relative monad

Flexibly graded clones

Explicitly, a flexibly graded clone T:

- maps $(d_1, \ldots, d_n)$ to a graded set $T(d_1, \ldots, d_n) : \mathbb{E} \to \mathbf{Set}$ (of terms)
- has tuples var $\in \prod_i T(d_1, \ldots, d_n) d_i$ (the variables)

  corresponding to $\eta : K(d_1, \ldots, d_n) \to T(d_1, \ldots, d_n)$
- has natural transformations (substitution)

$$\mathsf{subst} : T(d_1, \ldots, d_n) d' \times \prod_i T\,\Gamma\,(d_i \cdot e) \longrightarrow T\,\Gamma\,(d' \cdot e)$$

  corresponding to

$$(-)^\dagger : \mathbf{GSet}(K(d_1, \ldots, d_n), T\,\Gamma)\; e \to \mathbf{GSet}(T(d_1, \ldots, d_n), T\,\Gamma)\; e$$

Flexibly graded presentations

(Σ, E) consists of

- sets of operators $\mathrm{op} \in \Sigma(d_1, \ldots, d_n; d')$
- sets of equations $(t, u) \in E(d_1, \ldots, d_n; d')$ where $t, u \in \mathrm{Tm}_\Sigma(d_1, \ldots, d_n)d'$

with the sets $\mathrm{Tm}_\Sigma\,\Gamma\,d'$ of terms over Σ generated inductively by:

- $\mathrm{var}_i \in \mathrm{Tm}_\Sigma(d_1, \ldots, d_n)d_i$
  for each i
- $\mathrm{op}(e; t_1, \ldots, t_n) \in \mathrm{Tm}_\Sigma\,\Gamma\,(d' \cdot e)$
  for each $\mathrm{op} \in \Sigma(d_1, \ldots, d_n; d')$, $e \in |\mathbb{E}|$, $t \in \prod_i \mathrm{Tm}_\Sigma\,\Gamma\,(d_i \cdot e)$
- $\zeta^* t \in \mathrm{Tm}_\Sigma\,\Gamma\,d''$
  for each $t \in \mathrm{Tm}_\Sigma\,\Gamma\,d'$, $\zeta \in \mathbb{E}(d', d'')$

E induces an equivalence relation $\equiv$ on terms, and

$$\mathrm{Tm}_{(\Sigma,E)}(d_1, \ldots, d_n)d' \;=\; \mathrm{Tm}_{(\Sigma,E)}(d_1, \ldots, d_n)d' \,/ \equiv$$

forms a flexibly graded clone $\mathrm{Tm}_{(\Sigma,E)}$

Presenting graded monoids

Grades:

$$\mathbb{E} = (\mathbb{N}_{\leq}, 1, \cdot)$$

Operators:

$$u \in \Sigma(\ ;\ 0) \qquad m_{d_1,d_2} \in \Sigma(d_1, d_2\ ;\ (d_1 + d_2)) \qquad \text{(for each } d_1, d_2 \in \mathbb{N}\text{)}$$

Equations:

$$\begin{aligned}
m_{0,d}(1; u, \mathsf{var}_1) &= \mathsf{var}_1 \\
\mathsf{var}_1 &= m_{d,0}(1; \mathsf{var}_1, u) \\
m_{d_1+d_2,d_3}(1; m_{d_1,d_2}(1; \mathsf{var}_1, \mathsf{var}_2), \mathsf{var}_3) &= m_{d_1,d_2+d_3}(1; \mathsf{var}_1, m_{d_2,d_3}(1; \mathsf{var}_2, \mathsf{var}_3)) \\
m_{d_1',d_2'}(1; (d_1 \leq d_1')^* \mathsf{var}_1, (d_2 \leq d_2')^* \mathsf{var}_2) &= ((d_1 + d_2) \leq (d_1' + d_2'))^* (m_{d_1,d_2}(1; \mathsf{var}_1, \mathsf{var}_2)) \\
m_{d_1,d_2}(d; \mathsf{var}_1, \mathsf{var}_2) &= m_{d_1 \cdot e, d_2 \cdot e}(1; \mathsf{var}_1, \mathsf{var}_2)
\end{aligned}$$

There is an equivalence

$$\text{flexibly graded presentations} \quad \underset{(\Sigma_{\mathsf{T}}, E_{\mathsf{T}}) \leftarrow\!\shortmid \mathsf{T}}{\overset{\mathsf{Tm}_{(\Sigma,E)}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \quad \text{flexibly graded clones}$$

satisfying

$$\begin{tikzcd}
\mathbf{Alg}(\Sigma, E) \arrow[rr, "\cong"] \arrow[dr, "U_{(\Sigma,E)}"'] & & \mathbf{EM}(\mathsf{Tm}_{(\Sigma,E)}) \arrow[dl, "U_{\mathsf{Tm}_{(\Sigma,E)}}"] \\
& \mathbf{GSet} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Alg}(\Sigma_{\mathsf{T}}, E_{\mathsf{T}}) \arrow[rr, "\cong"] \arrow[dr, "U_{(\Sigma_{\mathsf{T}},E_{\mathsf{T}})}"'] & & \mathbf{EM}(\mathsf{T}) \arrow[dl, "U_{\mathsf{T}}"] \\
& \mathbf{GSet} &
\end{tikzcd}$$

K as a cocompletion

- A small category $\mathbb{I}$ is sifted when $\mathbb{I}$-colimits commute with finite products in $\mathbf{Set}$
- A conical sifted colimit in an $[\mathbb{E}, \mathbf{Set}]$-category C is a conical colimit of a sifted diagram $\mathbb{I} \longrightarrow \underline{C}$

$K : \mathbf{FCtx} \longrightarrow \mathbf{GSet}$ is the free completion of $\mathbf{FCtx}$ under conical sifted colimits:
If C has conical sifted colimits, then

$$\begin{tikzcd} \begin{array}{c}[\mathbb{E}, \mathbf{Set}]\text{-functors} \\ \mathbf{FCtx} \longrightarrow C\end{array} \arrow[r, shift left=3, "\mathrm{Lan}_K"] \arrow[r, phantom, "\simeq"] & \begin{array}{c}[\mathbb{E}, \mathbf{Set}]\text{-functors} \\ \mathbf{GSet} \longrightarrow C \\ \text{preserving conical sifted colimits}\end{array} \arrow[l, shift left=3, "- \circ K"] \end{tikzcd}$$

because

$$\mathrm{Lan}_K FX \cong \mathrm{Lan}_{\underline{K}} \underline{F} X \quad \text{and} \quad \underline{K} : \mathbb{E}^* \longrightarrow [\mathbb{E}, \mathbf{Set}] \text{ is a completion under sifted colimits}$$

The equivalence

$$\begin{array}{c}[\mathbb{E}, \mathbf{Set}]\text{-functors} \\ \mathbf{FCtx} \longrightarrow \mathbf{GSet}\end{array} \quad \underset{-\circ K}{\overset{\mathrm{Lan}_K}{\rightleftarrows}} \ \simeq \quad \begin{array}{c}[\mathbb{E}, \mathbf{Set}]\text{-functors} \\ \mathbf{GSet} \longrightarrow \mathbf{GSet} \\ \text{preserving conical sifted colimits}\end{array}$$

induces an equivalence

$$K\text{-relative monads} \quad \underset{-\circ K}{\overset{\mathrm{Lan}_K}{\rightleftarrows}} \ \simeq \quad \begin{array}{c}[\mathbb{E}, \mathbf{Set}]\text{-monads on } \mathbf{GSet} \\ \text{preserving conical sifted colimits}\end{array}$$

flexibly graded clones

satisfying

$$\begin{tikzcd} \mathbf{EM}(\mathsf{T}) \arrow[rr, "\cong"] \arrow[dr, "U_{\mathsf{T}}"'] & & \mathbf{EM}(\mathrm{Lan}_K \mathsf{T}) \arrow[dl, "U_{\mathrm{Lan}_K \mathsf{T}}"] \\ & \mathbf{GSet} & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{EM}(\mathsf{T}' \circ K) \arrow[rr, "\cong"] \arrow[dr, "U_{\mathsf{T}' \circ K}"'] & & \mathbf{EM}(\mathsf{T}') \arrow[dl, "U_{\mathsf{T}'}"] \\ & \mathbf{GSet} & \end{tikzcd}$$

Constructing a graded monad

There is an adjunction

$$\begin{tikzcd}[column sep=large] \begin{array}{c}[\mathbb{E}, \mathbf{Set}]\text{-monads on } \mathbf{GSet} \\ \text{preserving conical sifted colimits}\end{array} \arrow[r, shift left=2, "- \circ J"] \arrow[r, phantom, "\top"] & \begin{array}{c}(J : \mathbf{RSet} \to \mathbf{GSet})\text{-relative monads} \\ \text{preserving conical sifted colimits}\end{array} \arrow[l, shift left=2, "\lceil - \rceil"] \end{tikzcd}$$

graded monads

with a functor $R_{\mathsf{T}'} : \mathbf{EM}(\mathsf{T}') \to \mathbf{EM}(\mathsf{T}' \circ J)$ for each $[\mathbb{E}, \mathbf{Set}]$-monad T', satisfying

$$\begin{tikzcd} \mathbf{EM}(\mathsf{T}') \arrow[rr, "R_{\mathsf{T}'}"] \arrow[dr, swap, "U_{\mathsf{T}'}"] & & \mathbf{EM}(\mathsf{T}' \circ J) \arrow[dl, "U_{\mathsf{T}' \circ J}"] \\ & \mathbf{GSet} & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{EM}(\mathsf{T}'') \arrow[rr, "\cong"] \arrow[dr, swap, "U_{\mathsf{T}''}"] & & \mathbf{EM}(\lceil \mathsf{T}'' \rceil) \arrow[dl, "U_{\lceil \mathsf{T}'' \rceil}"] \\ & \mathbf{GSet} & \end{tikzcd}$$

$R_{\mathsf{T}'}$ is not in general an isomorphism

- there is no graded monad T'' such that $\mathbf{EM}(\mathsf{T}'') \cong \mathbf{GMon}$ over $\mathbf{GSet}$

Presenting graded monads

Theorem

For every flexibly graded presentation (Σ, E)*, there is*

- *a graded monad* $\mathsf{T}_{(\Sigma,E)}$
- *and functor* $R_{(\Sigma,E)} : \mathbf{Alg}(\Sigma, E) \to \mathbf{EM}(\mathsf{T}_{(\Sigma,E)})$ *over* $\mathbf{GSet}$

such that

- *for every graded monad* T' *and functor* $R' : \mathbf{Alg}(\Sigma, E) \to \mathbf{EM}(\mathsf{T}')$ *over* $\mathbf{GSet}$*, there is a unique* $\alpha : \mathsf{T}' \to \mathsf{T}_{(\Sigma,E)}$ *such that*

$$\begin{tikzcd} \mathbf{Alg}(\Sigma, E) \arrow[r, "R_{(\Sigma,E)}"] \arrow[dr, "R'"'] & \mathbf{EM}(\mathsf{T}_{(\Sigma,E)}) \arrow[d, "\mathbf{EM}(\alpha)"] \\ & \mathbf{EM}(\mathsf{T}') \end{tikzcd} \qquad \begin{tikzcd} \mathsf{T}_{(\Sigma,E)} \\ \mathsf{T}' \arrow[u, dashed, "\alpha"'] \end{tikzcd}$$

- *the free* $\mathsf{T}_{(\Sigma,E)}$*-algebra on a set* X *is the free* (Σ, E)*-algebra on* $\mathbb{E}(1, -) \bullet X$