Flexible presentations of graded monads

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Joint work with Shin-ya Katsumata, Tarmo Uustalu and Nicolas Wu
Motivation

1. Effects can be modelled using monads
   \[\text{[Moggi '89]}\]
2. which often come from presentations
   \[\text{[Plotkin and Power '02]}\]
3. which induce algebraic operations
   \[\text{[Plotkin and Power '03]}\]

Example:
1. Nondeterministic computations can be modelled using the free monoid monad List
2. which comes from the presentation of monoids
   \[\text{fail} : 0 \quad \text{or} : 2\]
   \[\text{or} (\text{fail}, x) = x = \text{or} (x, \text{fail}) \quad \text{or} (\text{or} (x, y), z) = \text{or} (x, \text{or} (y, z))\]
3. which induces algebraic operations
   \[\text{fail}_X = (\lambda_. \text{[]} : 1 \to \text{List} X\]
   \[\text{or}_X = (\lambda (xs, ys). xs ++ ys) : \text{List} X \times \text{List} X \to \text{List} X\]
Motivation

1. Effects with quantitative information can be modelled using graded monads \cite{Katsumata2014}

2. which often come from graded presentations? \cite{Smirnov2008,Milius2015,Dorsch2019,Kura2020}

3. which induce algebraic operations?
Goal

Develop a notion of flexibly graded presentation for graded monads

Each flexibly graded presentation \((\Sigma, E)\) induces

1. a flexibly graded (abstract) clone of terms
2. hence an \([E, \text{Set}]-\text{monad on GSet}\)
3. hence a graded monad
Graded monoids

A graded monoid $A$ is

- a functor $A : \mathbb{N}_{\leq} \to \text{Set}$
- with an element $u \in A_0$
- and a natural transformation $m_{d_1,d_2} : A d_1 \times A d_2 \to A(d_1 + d_2)$

such that

$$m_{0,d}(u,x) = x = m_{d,0}(x,u) \quad m_{d_1+d_2,d_3}(m_{d_1,d_2}(x,y),z) = m_{d_1,d_2+d_3}(x,m_{d_2,d_3}(y,z))$$
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A morphism $f : A \to A'$ of grade $e \in \mathbb{N}$ is a natural transformation

$$f : A \Rightarrow A'(- \cdot e)$$

preserving the structure:

$$f_0(u) = u' \quad f_{d_1+d_2}(m_{d_1,d_2}(x,y)) = m'_{d_1,d_2}(f_{d_1}(x),f_{d_2}(y))$$

So we get a $[\mathbb{N}_{\leq},\text{Set}]$-category $\text{GMon}$, and $U : \text{GMon} \to \text{GSet}$
Grading via \([\mathbb{E}, \text{Set}]\)-categories

Let \((\mathbb{E}, 1, \cdot)\) be a small strict monoidal category of grades

- for example \(\mathbb{N}_{\leq}\) with multiplication

- \([\mathbb{E}, \text{Set}]\) is a monoidal category with Day convolution

- we work in \([\mathbb{E}, \text{Set}]-\text{CAT}\)
Grading via \([\mathbb{E}, \text{Set}]-\text{categories}\)

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The \([\mathbb{E}, \text{Set}]-\text{category} \) \(\text{GSet}\) is \([\mathbb{E}, \text{Set}]\) enriched over itself:

- objects are graded sets \(X : \mathbb{E} \to \text{Set}\)
- morphisms of grade \(e\) (elements of \(\text{GSet}(X, Y)e\)) are natural transformations \(f : X \Rightarrow Y(\cdot \cdot e)\)

- identities \(\text{id}_X \in \text{GSet}(X, X)1\)
- composition

\[
(g \circ f) : X \Rightarrow Y(\cdot \cdot e) \xrightarrow{g \cdot e} Z(\cdot \cdot e \cdot e')
\]

where \(f \in \text{GSet}(X, Y)e\) and \(g \in \text{GSet}(Y, Z)e'\)
Relative monads in \([\mathbb{E}, \mathbb{Set}]\)-\text{CAT}

A \((J : \mathbb{A} \to \mathbb{C})\)-relative monad \(\mathbb{T}\) is:

1. a function \(T : |\mathbb{A}| \to |\mathbb{C}|\)
2. with a morphism \(\eta_X : JX \to TX\) for each \(X \in |\mathbb{A}|\)
3. and a natural transformation

\[
(-)^\dagger : C(JX, TY) \Rightarrow C(TX, TY)
\]

for each \(X, Y \in |\mathbb{A}|\)

such that

\[
f^\dagger \circ \eta_X = f \quad \eta_X^\dagger = id_X \quad (g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger
\]
Relative monads in $[\mathbb{E}, \text{Set}]$-$\text{CAT}$

A $(J : \mathcal{A} \to C)$-relative monad $T$ is:

- a function $T : |\mathcal{A}| \to |C|$
- with a morphism $\eta_X : JX \to TX$ for each $X \in |\mathcal{A}|$
- and a natural transformation

$$(\cdot)^\dagger : C(JX, TY) \Rightarrow C(TX, TY)$$

for each $X, Y \in |\mathcal{A}|$

such that

$$f^\dagger \circ \eta_X = f \quad \eta_X^\dagger = \text{id}_X \quad (g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger$$

$T$ has an Eilenberg-Moore $[\mathbb{E}, \text{Set}]$-category, and a forgetful $[\mathbb{E}, \text{Set}]$-functor

$U_T : \text{EM}(T) \to C$
Relative monads in $[\mathbb{E}, \text{Set}]$-CAT

A $(J : \mathcal{A} \to C)$-relative monad $T$ is: [Altenkirch, Chapman, Uustalu ’15]

- a function $T : |\mathcal{A}| \to |C|$
- with a morphism $\eta_X : JX \to TX$ for each $X \in |\mathcal{A}|$
- and a natural transformation

$(-) \dagger : C(JX, TY) \Rightarrow C(TX, TY)$

for each $X, Y \in |\mathcal{A}|$

such that

$f \dagger \circ \eta_X = f \quad \eta_X \dagger = \text{id}_X \quad (g \dagger \circ f) \dagger = g \dagger \circ f \dagger$

$T$ has an Eilenberg-Moore $[\mathbb{E}, \text{Set}]$-category, and a forgetful $[\mathbb{E}, \text{Set}]$-functor

$U_T : \text{EM}(T) \to C$

We rely heavily on some general results about relative monads (jww Nathanael Arkor)
Graded monads

For each set $X$, define

$$JX = \mathbb{E}(1, -) \bullet X : \mathbb{E} \to \text{Set}$$

so that

$$\text{GSet}(JX, A)e \cong \text{Set}(X, Ae)$$

and form a fully faithful $[\mathbb{E}, \text{Set}]$-functor

$$J : \text{RSet} \to \text{GSet}$$

($\text{RSet}$ is the free $[\mathbb{E}, \text{Set}]$-category on $\text{Set}$)

Definition: an $\mathbb{E}$-graded monad (on $\text{Set}$) is a $J$-relative monad

(This is equivalent to the definitions in [Smirnov '08, Melliès '12, Katsumata '14])
Flexibly graded clones

For each finite sequence $d_1, \ldots, d_n \in |\mathbb{E}|$, define

$$K(d_1, \ldots, d_n) = \prod_i \mathbb{E}(d_i, -) : \mathbb{E} \to \text{Set}$$

so that

$$\text{GSet}(K(d_1, \ldots, d_n), A)e \cong \prod_i A(d_i \cdot e)$$

and form a fully faithful $[\mathbb{E}, \text{Set}]$-functor

$$K : \text{FCtx} \to \text{GSet}$$

is a fully faithful $[\mathbb{E}, \text{Set}]$-functor

A flexibly graded (abstract) clone is a $K$-relative monad
Explicitly, a flexibly graded clone $T$:

- maps $(d_1, \ldots, d_n)$ to a graded set $T(d_1, \ldots, d_n) : \mathbb{E} \to \text{Set}$ (of terms)
- has tuples $\text{var} \in \prod_i T(d_1, \ldots, d_n) d_i$ (the variables) corresponding to $\eta : K(d_1, \ldots, d_n) \to T(d_1, \ldots, d_n)$
- has natural transformations (substitution)

$$\text{subst} : T(d_1, \ldots, d_n) d' \times \prod_i T \Gamma (d_i \cdot e) \to T \Gamma (d' \cdot e)$$

corresponding to

$$(-)^\dagger : \text{GSet}(K(d_1, \ldots, d_n), T \Gamma) e \to \text{GSet}(T(d_1, \ldots, d_n), T \Gamma) e$$
Flexibly graded presentations

\((\Sigma, E)\) consists of

- sets of operators \(\text{op} \in \Sigma(d_1, \ldots, d_n; d')\)
- sets of equations \((t, u) \in E(d_1, \ldots, d_n; d')\) where \(t, u \in Tm_\Sigma(d_1, \ldots, d_n)\)

with the sets \(Tm_\Sigma \Gamma d'\) of terms over \(\Sigma\) generated inductively by:

- \(\text{var}_i \in Tm_\Sigma(d_1, \ldots, d_n)d_i\) for each \(i\)
- \(\text{op}(e; t_1, \ldots, t_n) \in Tm_\Sigma \Gamma (d' \cdot e)\) for each \(\text{op} \in \Sigma(d_1, \ldots, d_n; d')\), \(e \in |E|\), \(t \in \prod_i Tm_\Sigma \Gamma (d_i \cdot e)\)
- \(\varphi^* t \in Tm_\Sigma \Gamma d''\) for each \(t \in Tm_\Sigma \Gamma d', \varphi \in \mathcal{E}(d', d'')\)

\(E\) induces an equivalence relation \(\equiv\) on terms, and

\[Tm_{(\Sigma, E)}(d_1, \ldots, d_n)d' = Tm_{(\Sigma, E)}(d_1, \ldots, d_n)d' / \equiv\]

forms a flexibly graded clone \(Tm_{(\Sigma, E)}\)
Presenting graded monoids

Grades:

\[ E = (\mathbb{N}_\leq, 1, \cdot) \]

Operators:

\[ u \in \Sigma(\ ; 0) \quad m_{d_1,d_2} \in \Sigma(d_1, d_2 ; (d_1 + d_2)) \quad \text{(for each } d_1, d_2 \in \mathbb{N}) \]

Equations:

\[ m_{0,d}(1; u, \text{var}_1) = \text{var}_1 \]
\[ \text{var}_1 = m_{d,0}(1; \text{var}_1, u) \]
\[ m_{d_1+d_2,d_3}(1; m_{d_1,d_2}(1; \text{var}_1, \text{var}_2), \text{var}_3) = m_{d_1,d_2+d_3}(1; \text{var}_1, m_{d_2,d_3}(1; \text{var}_2, \text{var}_3)) \]
\[ m_{d_1',d_2'}(1; (d_1 \leq d_1''){\text{var}_1}, (d_2 \leq d_2''){\text{var}_2}) = ((d_1 + d_2) \leq (d_1' + d_2'))^{\text{var}_1, \text{var}_2}) \]
\[ m_{d_1,d_2}(d; \text{var}_1, \text{var}_2) = m_{d_1\cdot e, d_2\cdot e}(1; \text{var}_1, \text{var}_2) \]
There is an equivalence

\[
\begin{align*}
\text{flexibly graded presentations} & \xrightarrow{Tm(\Sigma,E)} \xleftarrow{\approx} \text{flexibly graded clones} \\
\text{satisfying}
\end{align*}
\]

\[
\begin{align*}
\text{Alg}(\Sigma, E) & \xrightarrow{\approx} \text{EM}(Tm(\Sigma,E)) \\
\text{U}_{\Sigma,E} & \xrightarrow{} \text{GSet} \\
\text{Alg}(\Sigma_T, E_T) & \xrightarrow{\approx} \text{EM}(T) \\
\text{U}_{\Sigma_T,E_T} & \xrightarrow{} \text{GSet}
\end{align*}
\]
**$K$ as a cocompletion**

- A small category $\mathbb{I}$ is **sifted** when $\mathbb{I}$-colimits commute with finite products in Set.
- A **conical sifted colimit** in an $[\mathcal{E}, \text{Set}]$-category $C$ is a conical colimit of a sifted diagram $\mathbb{I} \to C$.

$K : \text{FCtx} \to \text{GSet}$ is the free completion of $\text{FCtx}$ under conical sifted colimits:

If $C$ has conical sifted colimits, then

$$[\mathcal{E}, \text{Set}]-\text{functors} \xrightarrow{\text{Lan}_K} [\mathcal{E}, \text{Set}]-\text{functors} \xrightarrow{\approx} \text{GSet} \to C$$

preserving conical sifted colimits

because

$$\text{Lan}_K FX \cong \text{Lan}_K F\mathcal{E} \quad \text{and} \quad K : \mathcal{E}^* \to [\mathcal{E}, \text{Set}] \text{ is a completion under sifted colimits}$$
The equivalence

\[ \mathcal{E}, \text{Set}\text{-}\text{functors} \quad \text{FCtx} \to \text{GSet} \xrightarrow{\text{Lan}_K} \mathcal{E}, \text{Set}\text{-}\text{functors} \quad \text{GSet} \to \text{GSet} \]

preserving conical sifted colimits

induces an equivalence

\[ \text{\textit{K}-relative monads} \xrightarrow{\text{Lan}_K} \text{\textit{K}-relative monads on GSet} \]

preserving conical sifted colimits

flexibly graded clones

satisfying

\[ \text{EM}(\mathcal{T}) \xrightarrow{\cong} \text{EM}(\text{Lan}_K \mathcal{T}) \]
\[ \text{EM}(\mathcal{T}') \circ K \xrightarrow{\cong} \text{EM}(\mathcal{T}') \]
Constructing a graded monad

There is an adjunction

\[
\begin{align*}
[\mathbb{E}, \text{Set}]\text{-monads on GSet} & \xrightarrow{\dashv} (J : \text{RSet} \to \text{GSet})\text{-relative monads} \\
preserving conical sifted colimits & \xleftarrow{\dashv} \text{preserving conical sifted colimits}
\end{align*}
\]

with a functor \( R_{T'} : \text{EM}(T') \to \text{EM}(T' \circ J) \) for each \([\mathbb{E}, \text{Set}]\text{-monad } T'\), satisfying

\[
\begin{align*}
\text{EM}(T') & \quad \xrightarrow{R_{T'}} \quad \text{EM}(T' \circ J) \\
\text{GSet} & \quad \xrightarrow{U_{T'}} \quad \text{GSet}
\end{align*}
\]

\[
\begin{align*}
\text{EM}(T'') & \quad \xrightarrow{\cong} \quad \text{EM}([T'']) \\
\text{GSet} & \quad \xrightarrow{U_{T''}} \quad \text{GSet}
\end{align*}
\]

\( R_{T'} \) is not in general an isomorphism

\( \triangleright \) there is no graded monad \( T'' \) such that \( \text{EM}(T'') \cong \text{GMon} \) over \( \text{GSet} \)
Theorem

For every flexibly graded presentation \((\Sigma, E)\), there is

- a graded monad \(T_{(\Sigma, E)}\)
- and functor \(R_{(\Sigma, E)} : \text{Alg}(\Sigma, E) \to \text{EM}(T_{(\Sigma, E)})\) over GSet

such that

- for every graded monad \(T'\) and functor \(R' : \text{Alg}(\Sigma, E) \to \text{EM}(T')\) over GSet, there is a unique \(\alpha : T' \to T_{(\Sigma, E)}\) such that

\[
\begin{array}{ccc}
\text{Alg}(\Sigma, E) & \xrightarrow{R_{(\Sigma, E)}} & \text{EM}(T_{(\Sigma, E)}) \\
& \downarrow{EM(\alpha)} & \uparrow{\alpha} \\
& \text{EM}(T') & T'
\end{array}
\]

- the free \(T_{(\Sigma, E)}\)-algebra on a set \(X\) is the free \((\Sigma, E)\)-algebra on \(B(1, -) \cdot X\)