Higher-order algebraic theories

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First-order theories have

$$\text{Operators}\ \frac{\Gamma \vdash t_1 \qquad \cdots \qquad \Gamma \vdash t_k}{\Gamma \vdash \mathsf{op}(t_1, \ldots, t_k)}\ (\mathsf{op} : k) \qquad \text{Equations}\ x_1, \ldots, x_k \vdash t \equiv u$$

Example: monoids have $\mathsf{e} : 0$, $\mathsf{mul} : 2$,

$$\frac{}{\Gamma \vdash \mathsf{e}} \quad \frac{\Gamma \vdash t_1 \qquad \Gamma \vdash t_2}{\Gamma \vdash \mathsf{mul}(t_1, t_2)} \qquad \begin{array}{rrcl} x \vdash & \mathsf{mul}(\mathsf{e}, x) & \equiv & x \\ x \vdash & x & \equiv & \mathsf{mul}(x, \mathsf{e}) \\ x_1, x_2, x_3 \vdash & \mathsf{mul}(\mathsf{mul}(x_1, x_2), x_3) & \equiv & \mathsf{mul}(x_1, \mathsf{mul}(x_2, x_3)) \end{array}$$

First-order theories have

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Example: monoids have $\mathsf{e} : 0$, $\mathsf{mul} : 2$,

$$\frac{}{\Gamma \vdash \mathsf{e}} \qquad \frac{\Gamma \vdash t_1 \qquad \Gamma \vdash t_2}{\Gamma \vdash \mathsf{mul}(t_1, t_2)}$$

$$\begin{aligned} x \vdash \quad & \mathsf{mul}(\mathsf{e}, x) & \equiv \quad & x \\ x \vdash \quad & x & \equiv \quad & \mathsf{mul}(x, \mathsf{e}) \\ x_1, x_2, x_3 \vdash \quad & \mathsf{mul}(\mathsf{mul}(x_1, x_2), x_3) & \equiv \quad & \mathsf{mul}(x_1, \mathsf{mul}(x_2, x_3)) \end{aligned}$$

Non-example: the untyped λ-calculus

$$\frac{\Gamma \vdash t_1 \qquad \Gamma \vdash t_2}{\Gamma \vdash \mathsf{app}(t_1, t_2)} \qquad \frac{\Gamma, x \vdash t}{\Gamma \vdash \mathsf{abs}(x.\, t)} \qquad \mathsf{app}(\mathsf{abs}(x.\, f), a) \equiv f[x \mapsto a]$$

First-order theories

- Presentations/equational logic
- Algebraic theories
- Finitary monads on $\mathbf{Set}$

Second-order theories: have variable-binding operators

- Presentations/equational logic [Fiore and Hur '10]
- Algebraic theories [Fiore and Mahmoud '10]

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This talk:

1. nth-order presentations
2. nth-order algebraic theories
3. a monad–theory correspondence

$(n \in \mathbb{N} \cup \{\omega\})$

First-order presentations

A (monosorted) first-order presentation is a signature with a set of equations, where:

- First-order arities are natural numbers k
- Signatures Σ are families of sets $\Sigma(k)$ of k-ary operators
- Contexts $\Gamma = x_1, \ldots, x_n$ are lists of variables
- Terms t are generated by

$$\frac{x \in \Gamma}{\Gamma \vdash x} \qquad\qquad \frac{\mathsf{op} \in \Sigma(k) \qquad \Gamma \vdash t_1 \qquad \cdots \qquad \Gamma \vdash t_k}{\Gamma \vdash \mathsf{op}(t_1, \ldots, t_k)}$$

- Equations $\Gamma \vdash t \equiv t'$

Example: monoids have $\mathsf{e} \in \Sigma(0)$, $\mathsf{mul} \in \Sigma(2)$,

$$\frac{}{\Gamma \vdash \mathsf{e}} \qquad \frac{\Gamma \vdash t_1 \qquad \Gamma \vdash t_2}{\Gamma \vdash \mathsf{mul}(t_1, t_2)} \qquad \begin{array}{rrcl} x \vdash & \mathsf{mul}(\mathsf{e}, x) & \equiv & x \\ x \vdash & x & \equiv & \mathsf{mul}(x, \mathsf{e}) \\ x_1, x_2, x_3 \vdash & \mathsf{mul}(\mathsf{mul}(x_1, x_2), x_3) & \equiv & \mathsf{mul}(x_1, \mathsf{mul}(x_2, x_3)) \end{array}$$

First-order presentations

For STLC with a base type s and operators $\mathsf{op} \in \Sigma$, terms

$$x_1 : s, \ldots, x_n : s \vdash t : s$$

have η-long β-normal forms generated by

$$\frac{(x : s) \in \Gamma}{\Gamma \vdash x : s} \qquad\qquad \frac{\mathsf{op} \in \Sigma(k) \qquad \Gamma \vdash t_1 : s \qquad \cdots \qquad \Gamma \vdash t_k : s}{\Gamma \vdash \mathsf{op}(t_1, \ldots, t_k) : s}$$

Second-order presentations [Fiore and Hur '10, Fiore and Mahmoud '10]

A (monosorted) second-order presentation is a signature with a set of equations, where:

- Second-order arities are lists $(n_1, \ldots, n_k)$ of natural numbers
- Signatures Σ are families of sets $\Sigma(n_1, \ldots, n_k)$ of $(n_1, \ldots, n_k)$-ary operators
- Variable contexts Γ and metavariable contexts Θ:

$$\Gamma = x_1, \ldots, x_n \qquad \Theta = \alpha_1 : m_1, \ldots, \alpha_p : m_p$$

- Terms t are generated by

$$\frac{x \in \Gamma}{\Theta \mid \Gamma \vdash x} \qquad \frac{(\alpha : m) \in \Theta \qquad \Theta \mid \Gamma \vdash t_1 \qquad \Theta \mid \Gamma \vdash t_m}{\Theta \mid \Gamma \vdash \alpha(t_1, \ldots, t_m)}$$

$$\frac{\begin{array}{c}(\mathsf{op} : (n_1, \ldots, n_k)) \in \Sigma \\ \Theta \mid \Gamma, x_{11}, \ldots, x_{1n_1} \vdash t_1 \qquad \cdots \qquad \Theta \mid \Gamma, x_{1k}, \ldots, x_{kn_k} \vdash t_k\end{array}}{\Theta \mid \Gamma \vdash \mathsf{op}(\vec{x_1}.\, t_1, \ldots, \vec{x_n}.\, t_k)}$$

- Equations $\Theta \mid \Gamma \vdash t \equiv t'$

Second-order presentations [Fiore and Hur '10, Fiore and Mahmoud '10]

$$\frac{(\mathsf{op} : (n_1, \ldots, n_k)) \in \Sigma \qquad \Theta \mid \Gamma, x_{11}, \ldots, x_{1n_1} \vdash t_1 \qquad \cdots \qquad \Theta \mid \Gamma, x_{1k}, \ldots, x_{kn_k} \vdash t_n}{\Theta \mid \Gamma \vdash \mathsf{op}(\vec{x_1}.\, t_1, \ldots, \vec{x_n}.\, t_k)}$$

Example: untyped λ-calculus has operators app $\in \Sigma(0,0)$ and abs $\in \Sigma(1)$

$$\frac{\Theta \mid \Gamma \vdash t_1 \qquad \Theta \mid \Gamma \vdash t_2}{\Theta \mid \Gamma \vdash \mathsf{app}(t_1, t_2)} \qquad\qquad \frac{\Theta \mid \Gamma, x \vdash t}{\Theta \mid \Gamma \vdash \mathsf{abs}(x.\, t)}$$

and equations

$$\alpha_1 : 1, \alpha_2 : 0 \mid \diamond \vdash \mathsf{app}(\mathsf{abs}(x.\, \alpha_1(x)), \alpha_2()) \;\equiv\; \alpha_1(\alpha_2()) \tag{β}$$

$$\alpha : 0 \mid \diamond \vdash \mathsf{abs}(x.\, \mathsf{app}(\alpha(), x)) \;\equiv\; \alpha() \tag{η}$$

Second-order presentations [Fiore and Hur '10, Fiore and Mahmoud '10]

Normal forms of STLC terms

$$\alpha_1 : (s^{m_1} \Rightarrow s), \ldots, \alpha_p : (s^{m_p} \Rightarrow s) \vdash t : s^n \Rightarrow s$$

with

$$\frac{(\mathsf{op} : (n_1, \ldots, n_k)) \in \Sigma \qquad \Gamma \vdash t_1 : s^{n_1} \Rightarrow s \qquad \cdots \qquad \Gamma \vdash t_k : s^{n_k} \Rightarrow s}{\Gamma \vdash \mathsf{op}(t_1, \ldots, t_k) : s}$$

are in bijection with terms

$$\alpha_1 : m_1, \ldots, \alpha_p : m_p \mid x_1, \ldots, x_n \vdash t$$

generated by

$$\frac{x \in \Gamma}{\Theta \mid \Gamma \vdash x} \qquad\qquad \frac{(\alpha : m) \in \Theta \qquad \Theta \mid \Gamma \vdash t_1 \qquad \Theta \mid \Gamma \vdash t_m}{\Theta \mid \Gamma \vdash \alpha(t_1, \ldots, t_m)}$$

$$\frac{(\mathsf{op} : (n_1, \ldots, n_k)) \in \Sigma \qquad \Theta \mid \Gamma, x_{11}, \ldots, x_{1n_1} \vdash t_1 \qquad \cdots \qquad \Theta \mid \Gamma, x_{1k}, \ldots, x_{kn_k} \vdash t_n}{\Theta \mid \Gamma \vdash \mathsf{op}(\vec{x_1}.\, t_1, \ldots, \vec{x_n}.\, t_n)}$$

Moving to higher orders

Use part of STLC for the equational logic:

- First-order: no functions
- Second-order: only first-order functions
- order $(n+1)$: only nth-order functions
- order ω: all of STLC [Lambek and Scott '88]

Higher-order presentations

Fix a set S of sorts (base types) s

$$\begin{aligned} A, B ::= {} & s & \operatorname{ord} s &= 0 \\ \mid {} & 1 & \operatorname{ord} 1 &= -1 \\ \mid {} & A_1 \times A_2 & \operatorname{ord}(A_1 \times A_2) &= \max\{\operatorname{ord} A_1, \operatorname{ord} A_2\} \\ \mid {} & A \Rightarrow B & \operatorname{ord}(A \Rightarrow B) &= \max\{\operatorname{ord} A + 1, \operatorname{ord} B\} \end{aligned}$$

Definition

For $n \in \mathbb{N} \cup \{\omega\}$, an nth-order signature Σ consists of a set

$$\Sigma(A\,;\,s)$$

for each $s \in S$ and A such that $\operatorname{ord} A < n$

Example: untyped λ-calculus ($S = \{\mathsf{tm}\}$, $n = 2$)

$$\Sigma(\mathsf{tm} \times \mathsf{tm}\,;\,\mathsf{tm}) = \{\mathsf{app}\} \qquad \Sigma((\mathsf{tm} \Rightarrow \mathsf{tm})\,;\,\mathsf{tm}) = \{\mathsf{abs}\}$$

Higher-order presentations

Given an nth-order signature, generate STLC terms t with

$$\frac{\mathsf{op} \in \Sigma(A\,;\,s) \qquad \Gamma \vdash t : A}{\Gamma \vdash \mathsf{op}\,t : s}$$

Definition

An nth-order presentation consists of:

- An nth-order signature Σ
- A set of equations

$$x_1 : A_1, \ldots, x_k : A_k \vdash t \equiv u : s$$

such that $\max\{\mathrm{ord}\,A_1, \ldots, \mathrm{ord}\,A_k\} < n$.

Examples

Monoids are first-order, with $S = \{\mathsf{elem}\}$

- Operators

$$\mathsf{e} \in \Sigma(1\,;\,\mathsf{elem}) \qquad \mathsf{mul} \in \Sigma(\mathsf{elem} \times \mathsf{elem}\,;\,\mathsf{elem})$$

- Equations

$$x : \mathsf{elem} \vdash \mathsf{mul}(\mathsf{e}(), x) \equiv x : \mathsf{elem} \qquad x : \mathsf{elem} \vdash \mathsf{mul}(x, \mathsf{e}()) \equiv x : \mathsf{elem}$$

$$x_1 : \mathsf{elem}, x_2 : \mathsf{elem}, x_3 : \mathsf{elem} \;\vdash \mathsf{mul}(\mathsf{mul}(x_1, x_2), x_3) \;\equiv\; \mathsf{mul}(x_1, \mathsf{mul}(x_2, x_3)) \;:\; \mathsf{elem}$$

Examples

Untyped λ-calculus is second-order, with $S = \{\mathsf{tm}\}$

- Operators

$$\mathsf{app} \in \Sigma(\mathsf{tm} \times \mathsf{tm}\,;\,\mathsf{tm}) \qquad \mathsf{abs} \in \Sigma((\mathsf{tm} \Rightarrow \mathsf{tm})\,;\,\mathsf{tm})$$

- Equations

$$f : \mathsf{tm} \Rightarrow \mathsf{tm},\, x : \mathsf{tm} \vdash \quad \mathsf{app}\,(\mathsf{abs}(f), x) \equiv f\,x \quad : \mathsf{tm} \quad (\beta)$$

$$f : \mathsf{tm} \vdash \quad \mathsf{abs}\,(\lambda x : \mathsf{tm}.\,\mathsf{app}\,(f, x)) \equiv f \quad : \mathsf{tm} \quad (\eta)$$

Examples

Simply typed λ-calculus is second-order, with $S = \{\mathsf{tm}_\tau \mid \tau \text{ is a type}\}$

$$\tau := \mathsf{b} \mid \tau \rightsquigarrow \tau'$$

- Operators

$$\mathsf{app}_{\tau,\tau'} \in \Sigma(\mathsf{tm}_{\tau \rightsquigarrow \tau'}, \mathsf{tm}_\tau \,;\, \mathsf{tm}_{\tau'}) \qquad \mathsf{abs}_{\tau,\tau'} \in \Sigma((\mathsf{tm}_\tau \Rightarrow \mathsf{tm}_{\tau'}) \,;\, \mathsf{tm}_{\tau \rightsquigarrow \tau'})$$

  for each τ, τ'

- Equations

$$\begin{aligned} f : \mathsf{tm}_\tau \Rightarrow \mathsf{tm}_{\tau'},\, x : \mathsf{tm}_\tau \vdash \quad & \mathsf{app}_{\tau,\tau'}(\mathsf{abs}_{\tau,\tau'}(f), x) \equiv f\,x && : \mathsf{tm}_{\tau'} && (\beta) \\ f : \mathsf{tm}_{\tau \rightsquigarrow \tau'} \vdash \quad & \mathsf{abs}_{\tau,\tau'}(\lambda x : \mathsf{tm}_\tau .\, \mathsf{app}_{\tau,\tau'}(f, x)) \equiv f && : \mathsf{tm}_{\tau \rightsquigarrow \tau'} && (\eta) \end{aligned}$$

  for each τ, τ'

Examples

Typed $\lambda\mu$-calculus is third-order [Abel '01], with sorts $S = \{\mathsf{tm}_\tau \mid \tau \text{ is a type}\} \cup \{\mathsf{nam}\}$

$$\begin{array}{rrcl}
\mathsf{app}_{\tau,\tau'} \in \Sigma(\mathsf{tm}_{\tau\rightsquigarrow\tau'}, \mathsf{tm}_\tau \,;\, \mathsf{tm}_{\tau'}) & \mathtt{t\,u} & \text{is} & \mathsf{app}_{\tau,\tau'}(t,u) \\
\mathsf{abs}_{\tau,\tau'} \in \Sigma((\mathsf{tm}_\tau \Rightarrow \mathsf{tm}_{\tau'}) \,;\, \mathsf{tm}_{\tau\rightsquigarrow\tau'}) & \lambda\mathtt{x} : \tau.\, \mathtt{t} & \text{is} & \mathsf{abs}_{\tau,\tau'}(\lambda x : \mathsf{tm}_\tau.\, t) \\
\mathsf{mu}_\tau \in \Sigma(((\mathsf{tm}_\tau \Rightarrow \mathsf{nam}) \Rightarrow \mathsf{nam}) \,;\, \mathsf{tm}_\tau) & \mu\alpha.\, \mathtt{t} & \text{is} & \mathsf{mu}_\tau(\lambda\alpha : \mathsf{tm}_\tau \Rightarrow \mathsf{nam}.\, t)
\end{array}$$

$$\begin{array}{lrcll}
\text{The named term} & [\alpha]\mathtt{t} & \text{is} & \alpha\, t & (\alpha : \mathsf{tm}_\tau \Rightarrow \mathsf{nam}, t : \mathsf{tm}_\tau) \\
\text{The mixed substitution} & \mathtt{u}[[\alpha](-) \mapsto \mathtt{v}(-)] & \text{is} & u\, v & \begin{array}{l}(u : (\mathsf{tm}_\tau \Rightarrow \mathsf{nam}) \Rightarrow \mathsf{nam}, \\ \; v : \mathsf{tm}_\tau \Rightarrow \mathsf{nam})\end{array}
\end{array}$$

A third-order equation:

$$\begin{aligned}
&\rho : (\mathsf{tm}_{\tau\rightsquigarrow\tau'} \Rightarrow \mathsf{nam}) \Rightarrow \mathsf{nam}, x : \mathsf{tm}_\tau \vdash \\
&\qquad \mathsf{app}_{\tau,\tau'}(\mathsf{mu}_{\tau\rightsquigarrow\tau'}(\rho), x) \\
&\qquad\quad \equiv \mathsf{mu}_{\tau'}(\lambda\beta : \mathsf{tm}_{\tau'} \Rightarrow \mathsf{nam}.\, \rho\, (\lambda f : \mathsf{tm}_{\tau\rightsquigarrow\tau'}.\, \beta\, (\mathsf{app}_{\tau,\tau'}(f, x)))) : \mathsf{tm}_{\tau'}
\end{aligned}$$

(which means $(\mu\alpha.\, \rho)\, \mathtt{x} \equiv \mu\beta.\, \rho[[\alpha](-) \mapsto [\beta]((-)\, \mathtt{x})]$)

Examples

Propositional logic/boolean algebras, with $S = \{\mathsf{prop}\}$

- Operators

$$\top \, \bot \in \Sigma(1\,;\,\mathsf{prop}) \qquad \text{(zeroth-order)}$$

$$\wedge \, \vee \in \Sigma(\mathsf{prop} \times \mathsf{prop}\,;\,\mathsf{prop}) \qquad \text{(first-order)}$$

$$\neg : \Sigma(\mathsf{prop}\,;\,\mathsf{prop}) \qquad \text{(first-order)}$$

- Many equations

Examples

First-order logic, with $S = \{\mathsf{prop}, \mathsf{thing}\}$

- Operators

$$
\begin{aligned}
\top\ \bot &\in \Sigma(1\,;\ \mathsf{prop}) && \text{(zeroth-order)}\\
\wedge\ \vee &\in \Sigma(\mathsf{prop} \times \mathsf{prop}\,;\ \mathsf{prop}) && \text{(first-order)}\\
\neg &: \Sigma(\mathsf{prop}\,;\ \mathsf{prop}) && \text{(first-order)}\\
\forall &\in \Sigma((\mathsf{thing} \Rightarrow \mathsf{prop})\,;\ \mathsf{prop}) && \text{(second-order)}
\end{aligned}
$$

- Many equations

Examples

Second-order logic, with $S = \{\mathsf{prop}, \mathsf{thing}\}$

- Operators

$$\begin{aligned}
\top\ \bot &\in \Sigma(1\,;\ \mathsf{prop}) && \text{(zeroth-order)}\\
\wedge\ \vee &\in \Sigma(\mathsf{prop}\times\mathsf{prop}\,;\ \mathsf{prop}) && \text{(first-order)}\\
\neg &: \Sigma(\mathsf{prop}\,;\ \mathsf{prop}) && \text{(first-order)}\\
\forall &\in \Sigma((\mathsf{thing}\Rightarrow\mathsf{prop})\,;\ \mathsf{prop}) && \text{(second-order)}\\
\forall_2 &: \Sigma((\mathsf{thing}\Rightarrow\mathsf{prop})\Rightarrow\mathsf{prop})\,;\ \mathsf{prop}) && \text{(third-order)}
\end{aligned}$$

- Many equations

Formula $\forall P.\forall x.\,(Px)\vee\neg(Px)$ encoded as

$$\forall_2\,(\lambda P:\mathsf{thing}\Rightarrow\mathsf{prop}.\,\forall\,(\lambda x:\mathsf{thing}.\,Px\vee\neg(Px)))$$

Examples

- Every parameterized algebraic theory [Staton '13] is a two-sorted second-order theory
- Partial differentiation has a monosorted second-order presentation [Plotkin '20]

nth-order presentations $\simeq$ nth-order algebraic theories

First-order algebraic theories

For $S = \{s\}$, first-order arities form a category $\mathcal{A}_1$, which:

- is the opposite of a skeleton of $\mathbf{FinSet}$
- is the free strict cartesian category on S
- has objects s^k for $k \in \mathbb{N}$, morphisms $t : s^k \to s^m$ are STLC terms

$$x : s^k \vdash t : s^m$$

  up to $\beta\eta$ (with no operators)

A first-order algebraic theory is a strict cartesian identity-on-objects functor

$$L : \mathcal{A}_1 \to \mathcal{L}$$

An element $t \in \mathcal{L}(s^k, s^m)$ “is” a term

$$x : s^k \vdash t : s^m$$

(possibly with operators, more equations)

Higher-order algebraic theories

Category of n-order arities $\mathcal{A}_n$, for $n \in \mathbb{N}_+ \cup \{\omega\}$:

- Objects are some representative subset of types A such that $\operatorname{ord} A < n$, with strict products and exponentials:

$$
\begin{gathered}
1 \times A \;=\; A \;=\; A \times 1 \qquad (A_1 \times A_2) \times A_3 \;=\; A_1 \times (A_2 \times A_3) \\
1 \Rightarrow A \;=\; A \qquad A \Rightarrow (A' \Rightarrow A'') \;=\; A \times A' \Rightarrow A'' \\
A \Rightarrow 1 \;=\; 1 \qquad A \Rightarrow (B_1 \times B_2) \;=\; (A \Rightarrow B_1) \times (A \Rightarrow B_2)
\end{gathered}
$$

- Morphisms $A \to B$ are STLC terms $x : A \vdash t : B$ up to $\beta\eta$

Some facts:

- $\mathcal{A}_{n+1}$ has exponentials $A \Rightarrow B$ for $A \in \mathcal{A}_n$, $B \in \mathcal{A}_{n+1}$
- $\mathcal{A}_{n+1}$ is the "free strict cartesian category on S in which S is exponentiable n times"
- for $n \leq n'$ there is a fully faithful functor $\mathcal{A}_n \hookrightarrow \mathcal{A}_{n'}$ (n'th-order STLC is a conservative extension of nth-order STLC)

Higher-order algebraic theories

Definition

For $n \in \mathbb{N}_+ \cup \{\omega\}$, an nth-order algebraic theory is a strict structure-preserving identity-on-objects functor

$$L : \mathcal{A}_n \to \mathcal{L}$$

Morphisms $F : L \to L'$ are commuting triangles

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "F"] & & \mathcal{L}' \\ & \mathcal{A}_n \arrow[ul, "L"] \arrow[ur, "L'"'] & \end{tikzcd}$$

Form a category $\mathbf{Law}_n$.

Theories from presentations

Given an nth-order presentation (Σ, E), have an nth-order algebraic theory $L : \mathcal{A}_n \to \mathcal{L}$:

- Objects of $\mathcal{L}$ are same as $\mathcal{A}_n$
- Morphisms $t : A \to B$ in $\mathcal{L}$ are terms

  $$x : A \vdash t : B$$

  over Σ, up to equivalence relation generated by E
- $L_{A,B} : \mathcal{A}_n(A, B) \to \mathcal{L}(A, B)$ is the inclusion

So we have:

$$\mathbf{Pres}_n \simeq \mathbf{Law}_n$$

Also for $n = 0$, where $\mathbf{Law}_0 = \mathbf{Set}^S$

A universal characterization of $\mathbf{Law}_n$

$\mathbf{Law}_n \simeq \mathbf{Cart}(\mathcal{A}_{n+1}, \mathbf{Set})$ for $n \in \mathbb{N}_+ \cup \{\omega\}$:

product-preserving functor $G : \mathcal{A}_{n+1} \to \mathbf{Set}$	$\mapsto$	nth-order algebraic theory $L_G : \mathcal{A}_n \to \mathcal{L}_G$ $\mathcal{L}_G(A, B) = G(A \Rightarrow B)$
nth-order algebraic theory $L : \mathcal{A}_n \to \mathcal{L}$	$\mapsto$	product-preserving functor $G_L : \mathcal{A}_{n+1} \to \mathbf{Set}$ $G_L(A \Rightarrow B) = \mathcal{L}(A, B)$

Also for $n = 0$:

$$\mathbf{Law}_0 = \mathbf{Set}^S \simeq \mathbf{Cart}(\mathcal{A}_1, \mathbf{Set})$$

A universal characterization of $\mathbf{Law}_n$

Since

$$\mathbf{Law}_n \simeq \mathbf{Cart}(\mathcal{A}_{n+1}, \mathbf{Set})$$

$\mathbf{Law}_n$ is the free completion of $\mathcal{A}_{n+1}^{\mathrm{op}}$ under sifted colimits:

$$\begin{array}{c} \text{functors} \\ \mathcal{A}_{n+1}^{\mathrm{op}} \to \mathcal{C} \end{array} \quad \underset{-\circ J_n}{\overset{\mathrm{Lan}_{J_n}}{\underset{\simeq}{\rightleftarrows}}} \quad \begin{array}{c} \text{sifted-colimit-preserving functors} \\ \mathbf{Law}_n \to \mathcal{C} \end{array}$$

when $\mathcal{C}$ has sifted colimits, where

$$J_n : \mathcal{A}_{n+1}^{\mathrm{op}} \xrightarrow{A \mapsto \mathcal{A}_{n+1}(A,-)} \mathbf{Cart}(\mathcal{A}_{n+1}, \mathbf{Set}) \simeq \mathbf{Law}_n$$

A universal characterization of $\mathbf{Law}_n$

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when $\mathcal{C}$ has sifted colimits, where

$$J_n : \mathcal{A}_{n+1}^{\mathrm{op}} \xrightarrow{A \mapsto \mathcal{A}_{n+1}(A,-)} \mathbf{Cart}(\mathcal{A}_{n+1}, \mathbf{Set}) \simeq \mathbf{Law}_n$$

So $\mathbf{Law}_n$ is:

- locally strongly finitely presentable
- complete and cocomplete

Semantics

An algebra of an $(n+1)$th-order algebraic theory

$$L : \mathcal{A}_{n+1} \to \mathcal{L}$$

is a cartesian functor

$$\mathcal{L} \to \mathbf{Set}$$

In terms of presentations, for $n \geq 1$:

- an nth-order algebraic theory L'
- with an interpretation

$$[\![\mathsf{op}]\!]_\Gamma : \prod_i \mathcal{L}'(\Gamma \times A_i, s_i) \to \mathcal{L}'(\Gamma, s')$$

  of each $\mathsf{op} \in \Sigma((A_1 \Rightarrow s_1) \times \cdots \times (A_k \Rightarrow s_k)\ ;\ s')$
- natural in $\Gamma \in \mathcal{A}_n$, and satisfying equations

Semantics

For the second-order presentation of STLC:

$$\begin{aligned}[\![\mathsf{app}_{\tau,\tau'}]\!]_\Gamma &: \mathcal{L}'(\Gamma, \mathsf{tm}_{\tau \rightsquigarrow \tau'}) \times \mathcal{L}'(\Gamma, \mathsf{tm}_\tau) \to \mathcal{L}'(\Gamma, \mathsf{tm}_{\tau'}) \\ [\![\mathsf{abs}_{\tau,\tau'}]\!]_\Gamma &: \mathcal{L}'(\Gamma \times \mathsf{tm}_\tau, \mathsf{tm}_{\tau'}) \to \mathcal{L}'(\Gamma, \mathsf{tm}_{\tau \rightsquigarrow \tau'})\end{aligned}$$

satisfying $\beta\eta$

For example:

$$\mathcal{L}'(\mathsf{tm}_{\tau_1} \times \cdots \times \mathsf{tm}_{\tau_k}, \mathsf{tm}_{\tau'}) = \text{STLC terms } x_1 : \tau_1, \ldots, x_k : \tau_k \vdash t : \tau' \text{ up to } \beta\eta$$

or

$$\mathcal{L}'(\mathsf{tm}_{\tau_1} \times \cdots \times \mathsf{tm}_{\tau_k}, \mathsf{tm}_{\tau'}) = \mathcal{C}(\textstyle\prod_i [\![\tau_i]\!], [\![\tau']\!])$$

for $\mathcal{C}$ a CCC with $[\![s]\!] \in |\mathcal{C}|$

Monad–theory correspondence

$(n+1)$th-order algebraic theories

$\simeq$ a class of monads on $\mathbf{Law}_n$

Monad–theory correspondence

$(n+1)$th-order algebraic theories

$\simeq$ a class of relative monads

$\simeq$ a class of monads on $\mathbf{Law}_n$

Theories from arities

There is a fully faithful functor

$$J_n : \mathcal{A}_{n+1}^{\mathrm{op}} \xrightarrow{X \mapsto \mathcal{A}_{n+1}(X,-)} \mathbf{Cart}(\mathcal{A}_{n+1}, \mathbf{Set}) \simeq \mathbf{Law}_n$$

- $J_n A$ is the nth-order theory $\mathcal{A}_n \to \mathcal{L}$ where

$$\mathcal{L}(B, B') = \mathcal{A}_{n+1}(A \times B, B')$$

- Objects $A \in \mathcal{A}_{n+1}$ correspond to finite nth-order signatures

$$\begin{pmatrix} (\mathsf{tm} \times \mathsf{tm} \Rightarrow \mathsf{tm}) \times \\ ((\mathsf{tm} \Rightarrow \mathsf{tm}) \Rightarrow \mathsf{tm}) \end{pmatrix} \in \mathcal{A}_3 \quad \text{corresponds to} \quad \begin{array}{c} \mathsf{app} \in \Sigma(\mathsf{tm} \times \mathsf{tm} \ ; \mathsf{tm}) \\ \mathsf{abs} \in \Sigma(\mathsf{tm} \Rightarrow \mathsf{tm} \ ; \mathsf{tm}) \end{array}$$

  and $J_n A$ is the theory presented by A with no equations

Relative monads [Altenkirch, Chapman, Uustalu '10]

Definition

A relative monad T on $J : \mathcal{A} \to \mathcal{C}$ consists of

- An object $T : |\mathcal{A}| \to |\mathcal{C}|$
- A morphism $\eta_X : JA \to TA$ for $A \in \mathcal{A}$
- Kleisli extension

$$\frac{f : JA \to TB}{f^\dagger : TA \to TB}$$

subject to laws: $f^\dagger \circ \eta_A = f \qquad {\eta_A}^\dagger = \mathrm{id}_{TA} \qquad (g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger$

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Each T has a Kleisli category $\mathbf{Kl}(\mathsf{T})$:

- Objects of $\mathbf{Kl}(\mathsf{T})$ are objects of $\mathcal{A}$
- Morphisms are given by $\mathbf{Kl}(\mathsf{T})(A, B) = \mathcal{C}(JA, TB)$

and a Kleisli inclusion

$$K_{\mathsf{T}} : \mathcal{A} \to \mathbf{Kl}(\mathsf{T}) \qquad K_{\mathsf{T}} A = A \qquad K_{\mathsf{T}} f = \eta_B \circ Jf$$

Theories from relative monads

If T is a relative monad on $J_n : \mathcal{A}_{n+1}^{\mathrm{op}} \to \mathbf{Law}_n$, then

$$K_{\mathsf{T}}^{\mathrm{op}} : \mathcal{A}_{n+1} \to (\mathbf{Kl}(\mathsf{T}))^{\mathrm{op}}$$

is an $(n+1)$th-order algebraic theory exactly when

$$TA + J_nB \cong T(A \times B) \qquad \text{(for all } A \in \mathcal{A}_{n+1},\ B \in \mathcal{A}_n\text{)}$$

Where $(L + J_nB) \in \mathbf{Law}_n$ is given for $B \in \mathcal{A}_n$ by

$$(\mathcal{L} + J_nB)(C, C') = \mathcal{L}(B \times C, C')$$

Relative monads from theories

Given an $(n+1)$th-order algebraic theory $L : \mathcal{A}_{n+1} \to \mathcal{L}$, define

$$\begin{array}{rcl} T_L : \mathcal{A}_{n+1}^{\mathrm{op}} & \to & \mathbf{Law}_n \\ T_L A\,(B, B') & = & \mathcal{L}(A \times B,\, B') \\ & \cong & \mathcal{L}(A,\, B \Rightarrow B') \end{array}$$

Then

$$\mathbf{Kl}(T_L)(B, A) \;=\; \mathbf{Law}_n(J_n B, T_L A) \;\cong\; \mathcal{L}(A, B)$$

so T_L forms a relative monad on $J_n : \mathcal{A}_{n+1}^{\mathrm{op}} \to \mathbf{Law}_n$:

$$\frac{\mathrm{id}_A : A \to A \text{ in } \mathcal{L}}{\eta_A : J_n A \to T_L A \text{ in } \mathbf{Law}_n}$$

$$\cfrac{\cfrac{f : J_n B \to T_L A \text{ in } \mathbf{Law}_n}{A \to B \text{ in } \mathcal{L}}}{f^\dagger : T_L B \to T_L A \text{ in } \mathbf{Law}_n}$$

A monad–theory correspondence

If $J : \mathcal{A} \to \mathcal{C}$ is a completion under Φ-colimits, then:

$$\begin{array}{c} \text{relative monads} \\ \text{on } J : \mathcal{A} \to \mathcal{C} \end{array} \quad \underset{-\circ J}{\overset{\mathrm{Lan}_J}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \quad \begin{array}{l} \Phi\text{-colimit-preserving} \\ \text{monads on } \mathcal{C} \end{array}$$

Theorem

There are equivalences between

- $(n+1)$*th-order algebraic theories*
- *Relative monads* T *on* $J_n : \mathcal{A}_{n+1}^{\mathrm{op}} \to \mathbf{Law}_n$ *such that*

$$TA + J_n B \cong T(A \times B) \qquad \text{(for all } A \in \mathcal{A}_{n+1},\ B \in \mathcal{A}_n)$$

- *Sifted-colimit-preserving monads* T *on* $\mathbf{Law}_n$ *such that*

$$TL + J_n B \cong T(L + J_n B) \qquad \text{(for all } L \in \mathbf{Law}_n,\ B \in \mathcal{A}_n)$$

For $n \in \mathbb{N} \cup \{\omega\}$, have notions of

- nth-order presentation
- nth-order algebraic theory

which:

- model syntax with variable binding operators
- are equivalent
- form locally strongly presentable categories
- correspond to a class of relative monads
- correspond to a class of monads
- have free algebras

(Slighly outdated) draft at `https://dylanm.org/drafts/hoat.pdf`

Coreflective subcategories of theories

Since

$$I_{n,n'} : \mathcal{A}_n \to \mathcal{A}_{n'} \qquad\qquad (n \le n')$$

is fully faithful and product-preserving, there are coreflections

$$\begin{tikzcd} \mathbf{Law}_n \arrow[r, hook, shift left=2, "\lceil - \rceil"] \arrow[r, phantom, "\perp"] & \mathbf{Law}_{n'} \arrow[l, shift left=2, "\lfloor - \rfloor"] \end{tikzcd}$$

Explicitly:

- $\lceil - \rceil : \mathbf{Law}_n \simeq \mathbf{Pres}_n \hookrightarrow \mathbf{Pres}_{n'} \simeq \mathbf{Law}_{n'}$
- For $n \geq 1$, if $L : \mathcal{A}_{n'} \to \mathcal{L}$ is an n'th-order algebraic theory, then

$$\lfloor L \rfloor : \mathcal{A}_n \to \lfloor \mathcal{L} \rfloor \qquad \lfloor \mathcal{L} \rfloor(A, B) = \mathcal{L}(A, B)$$