What makes a strong monad?

Dylan McDermott[1] Tarmo Uustalu[1,2]

[1] Reykjavik University, Iceland
[2] Tallinn University of Technology, Estonia

Strength

```
[1..3] >>= (\x ->
  [4..6] >>= (\y ->
    return (x + y)))
```

Since List is a monad

$$\frac{f : X \to \mathrm{List}Y}{{>\!\!>\!\!=}\, f : \mathrm{List}X \to \mathrm{List}Y}$$

we can interpret `>>= (\x -> ...)`:

$$\frac{(x \mapsto [x+4, x+5, x+6]) : \mathbb{Z} \to \mathrm{List}\,\mathbb{Z}}{{>\!\!>\!\!=}\,(x \mapsto [x+4, x+5, x+6]) : \mathrm{List}\,\mathbb{Z} \to \mathrm{List}\,\mathbb{Z}}$$

Strength

```
[1..3] >>= (\x ->
  [4..6] >>= (\y ->
    return (x + y)))
```

Since List is a strong monad

$$\frac{f : \Gamma \times X \to \mathrm{List}Y}{\ggg\!= f : \Gamma \times \mathrm{List}X \to \mathrm{List}Y}$$

we can interpret `>>= (\y -> ...)`:

$$\frac{((x, y) \mapsto [x + y]) : \mathbb{Z} \times \mathbb{Z} \to \mathrm{List}\mathbb{Z}}{\ggg\!= ((x, y) \mapsto [x + y]) : \mathbb{Z} \times \mathrm{List}\mathbb{Z} \to \mathrm{List}\mathbb{Z}}$$

This work

Collect together some useful results about strength:

- When do strengths exist?
- When are they unique?
- What about equivalent formulations?

Actions

An *action* of a monoidal category $(\mathbf{V}, I, \otimes)$ on a category $\mathbf{C}$ is:

- a bifunctor $\triangleright : \mathbf{V} \times \mathbf{C} \to \mathbf{C}$
- with isomorphisms

$$\lambda_X : I \triangleright X \cong X \qquad \alpha_{\Gamma',\Gamma,X} : (\Gamma' \otimes \Gamma) \triangleright X \cong \Gamma' \triangleright (\Gamma \triangleright X)$$

- satisfying coherence laws

Example: $(\mathbf{Set}, \mathbf{1}, \times)$ acts on $\mathbf{Set}$ with:

$$(\triangleright) = (\times) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set} \qquad \lambda(\star, x) = x \quad \alpha((\gamma', \gamma), x) = (\gamma', (\gamma, x))$$

Actions

An *action* of a monoidal category $(\mathbf{V}, I, \otimes)$ on a category $\mathbf{C}$ is:

- a bifunctor $\triangleright : \mathbf{V} \times \mathbf{C} \longrightarrow \mathbf{C}$
- with isomorphisms

$$\lambda_X : I \triangleright X \cong X \qquad \alpha_{\Gamma',\Gamma,X} : (\Gamma' \otimes \Gamma) \triangleright X \cong \Gamma' \triangleright (\Gamma \triangleright X)$$

- satisfying coherence laws

More generally: every cartesian monoidal $\mathbf{V}$ acts on itself with

$$(\triangleright) = (\times) : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}$$

- $\mathbf{Poset}$: posets and monotone functions
- $\mathbf{Set}_*$: pointed sets and point-preserving functions
- $[\mathbb{N}, \mathbf{Set}]$:
  - objects: pairs $(X, e : X \longrightarrow X)$,
  - morphisms: functions $f : X \longrightarrow Y$ such that $f(e_X x) = e_Y(fx)$
  - products: $X \times Y$ with $e_{X \times Y}(x, y) = (e_X x, e_Y y)$

Actions

An *action* of a monoidal category $(\mathbf{V}, I, \otimes)$ on a category $\mathbf{C}$ is:

- a bifunctor $\triangleright : \mathbf{V} \times \mathbf{C} \to \mathbf{C}$
- with isomorphisms

$$\lambda_X : I \triangleright X \cong X \qquad \alpha_{\Gamma',\Gamma,X} : (\Gamma' \otimes \Gamma) \triangleright X \cong \Gamma' \triangleright (\Gamma \triangleright X)$$

- satisfying coherence laws

Morphisms $\Gamma \triangleright X \to Y$ are "maps from X to Y in context Γ"

Identities:

$$\frac{}{\lambda_X : I \triangleright X \to X}$$

Composition:

$$\frac{f : \Gamma \triangleright X \to Y \qquad g : \Gamma' \triangleright Y \to Z}{(g \circ (\Gamma' \triangleright f) \circ \alpha) \ : \ (\Gamma' \otimes \Gamma) \triangleright X \to Z}$$

Strong functors

A strong functor $F : (\mathbf{C}, \triangleright_{\mathbf{C}}) \longrightarrow (\mathbf{D}, \triangleright_{\mathbf{D}})$ is ($\mathbf{V}$ acts on $\mathbf{C}$ and $\mathbf{D}$)

- an object assignment $F : |\mathbf{C}| \longrightarrow |\mathbf{D}|$
- a morphism assignment

$$\frac{f : \Gamma \triangleright_{\mathbf{C}} X \longrightarrow Y}{F^{(\Gamma)} f : \Gamma \triangleright_{\mathbf{D}} FX \longrightarrow FY}$$

- natural in Γ, and preserving identities and composition

Every strong functor induces an ordinary functor:

$$\begin{array}{c} X \longrightarrow Y \\ \hline\hline I \triangleright_{\mathbf{C}} X \longrightarrow Y \\ \hline I \triangleright_{\mathbf{D}} FX \longrightarrow FY \\ \hline\hline FX \longrightarrow FY \end{array}$$

Strong functors

A strong functor $F : (\mathbf{C}, \triangleright_{\mathbf{C}}) \longrightarrow (\mathbf{D}, \triangleright_{\mathbf{D}})$ is ($\mathbf{V}$ acts on $\mathbf{C}$ and $\mathbf{D}$)

- an object assignment $F : |\mathbf{C}| \longrightarrow |\mathbf{D}|$
- a morphism assignment

$$\frac{f : \Gamma \triangleright_{\mathbf{C}} X \longrightarrow Y}{F^{(\Gamma)} f : \Gamma \triangleright_{\mathbf{D}} FX \longrightarrow FY}$$

- natural in Γ, and preserving identities and composition

Every strong functor induces an ordinary functor

Example:

$$\mathrm{List} : (\mathbf{Set}, \times) \longrightarrow (\mathbf{Set}, \times)$$

$$\mathrm{List}\, X = \text{lists over } X$$

$$\mathrm{List}^{(\Gamma)} f(\gamma, [x_1, \ldots, x_n]) = [f(\gamma, x_1), \ldots, f(\gamma, x_n)]$$

Strong functors

A strong functor $F : (\mathbf{C}, \triangleright_{\mathbf{C}}) \to (\mathbf{D}, \triangleright_{\mathbf{D}})$ is (**V** acts on **C** and **D**)

- an object assignment $F : |\mathbf{C}| \to |\mathbf{D}|$
- a morphism assignment

$$\frac{f : \Gamma \triangleright_{\mathbf{C}} X \to Y}{F^{(\Gamma)} f : \Gamma \triangleright_{\mathbf{D}} FX \to FY}$$

- natural in Γ, and preserving identities and composition

Every strong functor induces an ordinary functor

Example:

$$\mathbb{N} \times - : ([\mathbb{N}, \mathbf{Set}], \times) \to ([\mathbb{N}, \mathbf{Set}], \times) \quad (\text{with } e(n, x) = (n, e\, x))$$

$$\mathbb{N} \times^{(\Gamma)} f : (\gamma, (n, x)) \mapsto (n, f(\gamma, x))$$

Strong functors

A strong functor $F : (\mathbf{C}, \triangleright_{\mathbf{C}}) \longrightarrow (\mathbf{D}, \triangleright_{\mathbf{D}})$ is (**V** acts on **C** and **D**)

- an object assignment $F : |\mathbf{C}| \longrightarrow |\mathbf{D}|$
- a morphism assignment

$$\frac{f : \Gamma \triangleright_{\mathbf{C}} X \longrightarrow Y}{F^{(\Gamma)} f : \Gamma \triangleright_{\mathbf{D}} FX \longrightarrow FY}$$

- natural in Γ, and preserving identities and composition

Every strong functor induces an ordinary functor

Example:

$$\mathbb{N} \times - : ([\mathbb{N}, \mathbf{Set}], \times) \longrightarrow ([\mathbb{N}, \mathbf{Set}], \times) \quad (\text{with } e(n, x) = (n, e\, x))$$

$$\mathbb{N} \times^{(\Gamma)} f : (\gamma, (n, x)) \mapsto (n, f(\gamma, x))$$

or:

$$\mathbb{N} \times^{(\Gamma)} f : (\gamma, (n, x)) \mapsto (n, f(e^n \gamma, x))$$

Strong functors

A strong functor $F : (\mathbf{C}, \triangleright_{\mathbf{C}}) \longrightarrow (\mathbf{D}, \triangleright_{\mathbf{D}})$ is (**V** acts on **C** and **D**)

- an object assignment $F : |\mathbf{C}| \longrightarrow |\mathbf{D}|$
- a morphism assignment

$$\frac{f : \Gamma \triangleright_{\mathbf{C}} X \longrightarrow Y}{F^{(\Gamma)} f : \Gamma \triangleright_{\mathbf{D}} FX \longrightarrow FY}$$

- natural in Γ, and preserving identities and composition

Equivalently:

- an ordinary functor $F : \mathbf{C} \longrightarrow \mathbf{D}$
- with a strength $\mathrm{str}_{\Gamma,X} : \Gamma \triangleright_{\mathbf{D}} FX \longrightarrow F(\Gamma \triangleright_{\mathbf{C}} X)$

  Example: $\mathbb{N} \times - : [\mathbb{N}, \mathbf{Set}] \longrightarrow [\mathbb{N}, \mathbf{Set}]$ with

  $\mathrm{str}(\gamma, (n, x)) = (n, (\gamma, x))$ or $\mathrm{str}(\gamma, (n, x)) = (n, (e^n \gamma, x))$

Uniqueness of strengths

Every morphism $f : \Gamma \triangleright_{\mathbf{D}} X \to Y$ can be applied at points of Γ:

$$\begin{aligned} (\!|f|\!) : \mathbf{V}(I, \Gamma) &\to \mathbf{D}(X, Y) \\ \gamma &\mapsto \left(X \xrightarrow{\lambda^{-1}} I \triangleright_{\mathbf{D}} X \xrightarrow{\gamma \triangleright_{\mathbf{D}} X} \Gamma \triangleright_{\mathbf{D}} X \xrightarrow{f} Y \right) \end{aligned}$$

Every strength for $F : \mathbf{C} \to \mathbf{D}$ satisfies

$$(\!|\mathrm{str}_{\Gamma,X}|\!)\gamma = F((\gamma \triangleright_{\mathbf{C}} X) \circ \lambda^{-1})$$

so if $\triangleright_{\mathbf{D}}$ is *well-pointed* ($(\!|-|\!)$ is injective), strengths are unique

Uniqueness of strengths

Every morphism $f : \Gamma \triangleright_{\mathbf{D}} X \to Y$ can be applied at points of Γ:

$$\begin{aligned} (\!| f |\!) : \mathbf{V}(I, \Gamma) &\to \mathbf{D}(X, Y) \\ \gamma &\mapsto \left(X \xrightarrow{\lambda^{-1}} I \triangleright_{\mathbf{D}} X \xrightarrow{\gamma \triangleright_{\mathbf{D}} X} \Gamma \triangleright_{\mathbf{D}} X \xrightarrow{f} Y \right) \end{aligned}$$

Every strength for $F : \mathbf{C} \to \mathbf{D}$ satisfies

$$(\!| \mathrm{str}_{\Gamma,X} |\!) \gamma = F((\gamma \triangleright_{\mathbf{C}} X) \circ \lambda^{-1})$$

so if $\triangleright_{\mathbf{D}}$ is *well-pointed* ($(\!| - |\!)$ is injective), strengths are unique

Example: every $F : \mathbf{Set} \to \mathbf{Set}$ has a unique strength

$$\mathrm{str}_{\Gamma,X}(\gamma, t) = F(x \mapsto (\gamma, x)) t$$

Uniqueness of strengths

Every morphism $f : \Gamma \triangleright_{\mathbf{D}} X \to Y$ can be applied at points of Γ:

$$\begin{aligned} (\!| f |\!) : \mathbf{V}(I, \Gamma) &\to \mathbf{D}(X, Y) \\ \gamma &\mapsto \left(X \xrightarrow{\lambda^{-1}} I \triangleright_{\mathbf{D}} X \xrightarrow{\gamma \triangleright_{\mathbf{D}} X} \Gamma \triangleright_{\mathbf{D}} X \xrightarrow{f} Y \right) \end{aligned}$$

Every strength for $F : \mathbf{C} \to \mathbf{D}$ satisfies

$$(\!| \mathrm{str}_{\Gamma, X} |\!) \gamma = F((\gamma \triangleright_{\mathbf{C}} X) \circ \lambda^{-1})$$

so if $\triangleright_{\mathbf{D}}$ is *well-pointed* ($(\!| - |\!)$ is injective), strengths are unique

Example: $|-| : (X, \leq) \mapsto (X, =) : \mathbf{Poset} \to \mathbf{Poset}$ has no strength:

$\mathrm{str}_{\Gamma, X} : (\gamma, x) \mapsto (\gamma, x) : \Gamma \times |X| \to |\Gamma \times X|$ is not monotone

Existence of strengths

If for every

$$\zeta : \mathbf{V}(I, \Gamma) \to \mathbf{D}(X, Y)$$

there is a

$$\Phi_\Gamma \zeta : \Gamma \triangleright_{\mathbf{D}} X \to Y$$

satisfying $(\!|\Phi_\Gamma \zeta|\!) = \zeta$ and respecting the action structure of $\triangleright_{\mathbf{D}}$, then

- every functor $F : \mathbf{C} \to \mathbf{D}$ forms a strong functor
- in a coherent way (natural transformations are strong)

Example: for $(\mathbf{D}, \triangleright) = (\mathbf{Set}, \times)$, take $\Phi_\Gamma \zeta(\gamma, x) = \zeta\, \gamma\, x$, then

$$F^{(\Gamma)} f\, (\gamma, t) = F(x \mapsto f(\gamma, x))\, t$$

Existence of strengths

If for every

$$\zeta : \mathbf{V}(I, \Gamma) \to \mathbf{D}(X, Y)$$

there is a

$$\Phi_\Gamma \zeta : \Gamma \triangleright_{\mathbf{D}} X \to Y$$

satisfying $(\!|\Phi_\Gamma \zeta|\!) = \zeta$ and respecting the action structure of $\triangleright_{\mathbf{D}}$, then

- every functor $F : \mathbf{C} \to \mathbf{D}$ forms a strong functor
- in a coherent way (natural transformations are strong)

Example: for $(\mathbf{D}, \triangleright) = (\mathbf{Set}_*, \times)$, take $\Phi_\Gamma \zeta(\gamma, x) = \zeta \star x$, then

$$F^{(\Gamma)} f\,(\gamma, t) = F(x \mapsto f(\star, x))\; t$$

(since $\mathbf{Set}_*(1, \Gamma) = \{\star\}$)

Strong monads

A strong monad on $(\mathbf{C}, \triangleright_{\mathbf{C}})$ is:

- an object mapping $X \mapsto TX$
- with unit morphisms $\eta_X : X \longrightarrow TX$
- and a Kleisli extension operation

$$\frac{f : \Gamma \triangleright X \longrightarrow TY}{f^{\dagger} : \Gamma \triangleright TX \longrightarrow TY}$$

- natural in Γ and satisfying three laws

Strong monads

A strong monad on $(\mathbf{C}, \triangleright_{\mathbf{C}})$ is:

- an object mapping $X \mapsto TX$
- with unit morphisms $\eta_X : X \longrightarrow TX$
- and a Kleisli extension operation

$$\frac{f : \Gamma \triangleright X \longrightarrow TY}{f^\dagger : \Gamma \triangleright TX \longrightarrow TY}$$

- natural in Γ and satisfying three laws

Example: take $(\mathbf{C}, \triangleright) = (\mathbf{Set}, \times)$ and

- $\mathrm{List}X$ = lists over X
- $\eta\, x = [x]$
- $f^\dagger(\gamma, [x_1, \ldots, x_n]) = f(\gamma, x_1) + \!\!\!+ \cdots + \!\!\!+ f(\gamma, x_n)$

Strong monads

A strong monad on $(\mathbf{C}, \triangleright_{\mathbf{C}})$ is:

- an object mapping $X \mapsto TX$
- with unit morphisms $\eta_X : X \longrightarrow TX$
- and a Kleisli extension operation

$$\frac{f : \Gamma \triangleright X \longrightarrow TY}{f^\dagger : \Gamma \triangleright TX \longrightarrow TY}$$

- natural in Γ and satisfying three laws

Example: take $(\mathbf{C}, \triangleright) = ([\mathbb{N}, \mathbf{Set}], \times)$ and

- $X \mapsto \mathbb{N} \times X$ (with $e(n, x) = (n, e\,x)$)
- $\eta\, x = (0, x)$
- $f^\dagger(\gamma, (n, x)) = (n + m, y)$ where $(m, y) = f(\gamma, x)$
  -or-
  $f^\dagger(\gamma, (n, x)) = (n + m, y)$ where $(m, y) = f(e^n \gamma, x)$

Strong monads

A strong monad on $(\mathbf{C}, \triangleright_{\mathbf{C}})$ is:

- an object mapping $X \mapsto TX$
- with unit morphisms $\eta_X : X \longrightarrow TX$
- and a Kleisli extension operation

$$\frac{f : \Gamma \triangleright X \longrightarrow TY}{f^{\dagger} : \Gamma \triangleright TX \longrightarrow TY}$$

- natural in Γ and satisfying three laws

Equivalently:

- A strong functor T, with a strong unit and a strong multiplication, satisfying the monad laws
- A monad (T, η, μ) with a lifting of $\triangleright$ to $\mathbf{Kl}\,\mathsf{T}$

$$\begin{array}{ccc} \mathbf{V} \times \mathbf{C} & \xrightarrow{\triangleright} & \mathbf{C} \\ {\scriptstyle \mathbf{V} \times K_{\mathsf{T}}} \big\downarrow & & \big\downarrow {\scriptstyle K_{\mathsf{T}}} \\ \mathbf{V} \times \mathbf{Kl}\,\mathsf{T} & \xrightarrow[\triangleright_{\mathsf{T}}]{} & \mathbf{Kl}\,\mathsf{T} \end{array}$$

Uniqueness and existence of strengths

- If $\triangleright$ is well-pointed, then strengths are unique

$$(\!|\mathrm{str}_{\Gamma,X}|\!)\gamma = T((\gamma \triangleright_{\mathbf{C}} X) \circ \lambda^{-1})$$

- Existence doesn't work as well: Φ makes Id into a strong monad only if $\triangleright$ is well-pointed
  - On $\mathbf{Set}_*$, defining $f^\dagger(\gamma, x) = f(\star, x)$ does not make Id into a strong monad

Equivalent perspectives: enrichment

For each $\mathbf{C}$, if $- \triangleright X \dashv X \multimap - : \mathbf{C} \to \mathbf{V}$ for each X:

- to make $\triangleright$ into an action is equivalently
- to make $\multimap$ into an enrichment of $\mathbf{C}$ over $\mathbf{V}$ such that $(\Gamma \triangleright X) \multimap Y \cong \Gamma \multimap (X \multimap Y)$

(when $\mathbf{V}$ is left closed)

Equivalent perspectives: enrichment

For each $\mathbf{C}$, if $- \triangleright X \dashv X \multimap\!\triangleright - : \mathbf{C} \to \mathbf{V}$ for each X:

- to make $\triangleright$ into an action is equivalently
- to make $\multimap\!\triangleright$ into an enrichment of $\mathbf{C}$ over $\mathbf{V}$ such that $(\Gamma \triangleright X) \multimap\!\triangleright Y \cong \Gamma \multimap (X \multimap\!\triangleright Y)$

(when $\mathbf{V}$ is left closed)

Under this bijection:

- strong functors are the same as enriched functors

$$X \mapsto FX$$

$$\mathrm{fmap}_{X,Y} : (X \multimap\!\triangleright Y) \to (FX \multimap\!\triangleright FY)$$

- strong monads are the same as enriched monads

$$X \mapsto TX$$

$$\eta_X : X \to TX$$

$$\mathrm{bind}_{X,Y} : (X \multimap\!\triangleright TY) \to (TX \multimap\!\triangleright TY)$$

Equivalent perspectives: enrichment

For each $\mathbf{C}$, if $- \triangleright X \dashv X \mathrel{-\!\!\triangleright} - : \mathbf{C} \to \mathbf{V}$ for each X:

- to make $\triangleright$ into an action is equivalently
- to make $\mathrel{-\!\!\triangleright}$ into an enrichment of $\mathbf{C}$ over $\mathbf{V}$ such that $(\Gamma \triangleright X) \mathrel{-\!\!\triangleright} Y \cong \Gamma \multimap (X \mathrel{-\!\!\triangleright} Y)$

(when $\mathbf{V}$ is left closed)

Under this bijection:

- strong functors are the same as enriched functors

$$X \mapsto FX$$
$$\mathrm{fmap}_{X,Y} : (X \mathrel{-\!\!\triangleright} Y) \to (FX \mathrel{-\!\!\triangleright} FY)$$

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

- strong monads are the same as enriched monads

$$X \mapsto TX$$
$$\eta_X : X \to TX$$
$$\mathrm{bind}_{X,Y} : (X \mathrel{-\!\!\triangleright} TY) \to (TX \mathrel{-\!\!\triangleright} TY)$$

```
class Monad t where
  return :: a -> t a
  (>>=) :: (a -> t b) -> t a -> t b
```

- There are many different ways of formulating strength
- arising by looking at strength from different perspectives
- leading to various different properties (existence, uniqueness, etc.)

Some other things (in the paper):

- Third perspective: powering $\Gamma \triangleright - \dashv \Gamma \pitchfork - : \mathbf{C} \to \mathbf{C}$
  $\rightsquigarrow$ formulation of strength in terms of $\mathbf{Alg}\,(T)$
  $\rightsquigarrow$ strengths for free monads
- Bistrengths and commutative monads