What makes a strong monad?

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Since List is a monad

\[ f : X \rightarrow \text{List} Y \]

\[ \gg= f : \text{List} X \rightarrow \text{List} Y \]

we can interpret \( \gg= (\backslash x \rightarrow \ldots ) : \)

\[ (x \mapsto [x + 4, x + 5, x + 6]) : \mathbb{Z} \rightarrow \text{List} \mathbb{Z} \]

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**Strength**

\[ [1..3] \gg= (x \rightarrow [4..6] \gg= (y \rightarrow \text{return } (x + y))) \]

Since List is a strong monad

\[ f : \Gamma \times X \rightarrow \text{ListY} \]

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we can interpret \( \gg= (y \rightarrow \ldots) : \)

\[ ((x, y) \mapsto [x + y]) : \mathbb{Z} \times \mathbb{Z} \rightarrow \text{List}\mathbb{Z} \]

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This work

Collect together some useful results about strength:

- When do strengths exist?
- When are they unique?
- What about equivalent formulations?
Actions

An action of a monoidal category \((V, I, \otimes)\) on a category \(C\) is:

- a bifunctor \(\triangleright : V \times C \rightarrow C\)
- with isomorphisms

\[
\lambda_X : I \triangleright X \cong X \quad \alpha_{\Gamma', \Gamma, X} : (\Gamma' \otimes \Gamma) \triangleright X \cong \Gamma' \triangleright (\Gamma \triangleright X)
\]

- satisfying coherence laws

Example: \((\text{Set}, 1, \times)\) acts on \(\text{Set}\) with:

\[
(\triangleright) = (\times) : \text{Set} \times \text{Set} \rightarrow \text{Set} \quad \lambda(\star, x) = x \quad \alpha((y', y), x) = (y', (y, x))
\]
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More generally: every cartesian monoidal \(V\) acts on itself with

\[
(\triangleright) = (\times) : V \times V \to V
\]

- **Poset**: posets and monotone functions
- **Set\(^*\)**: pointed sets and point-preserving functions
- **[\(\mathbb{N}, \text{Set}\)]**:
  - objects: pairs \((X, e : X \to X)\),
  - morphisms: functions \(f : X \to Y\) such that \(f(e_X x) = e_Y(f x)\)
  - products: \(X \times Y\) with \(e_{X \times Y}(x, y) = (e_X x, e_Y y)\)
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  \]
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Morphisms \(\Gamma \triangleright X \to Y\) are “maps from \(X\) to \(Y\) in context \(\Gamma\)”

Identities:

\[
\lambda_X : I \triangleright X \to X
\]

Composition:

\[
f : \Gamma \triangleright X \to Y \quad g : \Gamma' \triangleright Y \to Z
\]

\[
(g \circ (\Gamma' \triangleright f) \circ \alpha) : (\Gamma' \otimes \Gamma) \triangleright X \to Z
\]
Strong functors

A strong functor $F : (C, \triangleright_C) \to (D, \triangleright_D)$ is

- an object assignment $F : |C| \to |D|$ (V acts on C and D)
- a morphism assignment

\[
\frac{f : \Gamma \triangleright_C X \to Y}{F(\Gamma)f : \Gamma \triangleright_D FX \to FY}
\]

- natural in $\Gamma$, and preserving identities and composition

Every strong functor induces an ordinary functor:

\[
\frac{X \to Y}{I \triangleright_C X \to Y}
\]

\[
\frac{I \triangleright_D FX \to FY}{FX \to FY}
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Every strong functor induces an ordinary functor

Example:

List : $(\text{Set}, \times) \to (\text{Set}, \times)$

List $X = \text{lists over } X$

\[
\begin{align*}
    \text{List}^{(\Gamma)} f (\gamma, [x_1, \ldots, x_n]) & = [f (\gamma, x_1), \ldots, f (\gamma, x_n)]
\end{align*}
\]
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Every strong functor induces an ordinary functor

Example:

$$\mathbb{N} \times - : ([\mathbb{N}, \text{Set}], \times) \to ([\mathbb{N}, \text{Set}], \times) \quad (\text{with } e(n, x) = (n, e \times))$$

$$\mathbb{N} \times^{(\Gamma)} f : (\gamma, (n, x)) \mapsto (n, f(\gamma, x))$$
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$$\mathbb{N} \times^{(\Gamma)} f : (\gamma, (n, x)) \mapsto (n, f(\gamma, x))$$

or:

$$\mathbb{N} \times^{(\Gamma)} f : (\gamma, (n, x)) \mapsto (n, f(e^n \gamma, x))$$
Strong functors

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- natural in $\Gamma$, and preserving identities and composition

Equivalently:

- an ordinary functor $F : C \to D$
- with a strength $\text{str}_{\Gamma, X} : \Gamma \triangleright_D FX \to F(\Gamma \triangleright_C X)$

Example: $\mathbb{N} \times - : [\mathbb{N}, \text{Set}] \to [\mathbb{N}, \text{Set}]$ with $\text{str}(\gamma, (n, x)) = (n, (\gamma, x))$ or $\text{str}(\gamma, (n, x)) = (n, (e^n \gamma, x))$
Uniqueness of strengths

Every morphism \( f : \Gamma \triangleright_D X \to Y \) can be applied at points of \( \Gamma \):

\[
\langle \| f \| \rangle : V(I, \Gamma) \to D(X, Y)
\]

\[
\gamma \mapsto \left( X \xrightarrow{\lambda^{-1}} I \triangleright_D X \xrightarrow{\gamma \triangleright_D X} \Gamma \triangleright_D X \xrightarrow{f} Y \right)
\]

Every strength for \( F : C \to D \) satisfies

\[
\langle \| \text{str}_{\Gamma,X} \| \rangle \gamma = F((\gamma \triangleright_C X) \circ \lambda^{-1})
\]

so if \( \triangleright_D \) is well-pointed (\( \langle \| - \| \rangle \) is injective), strengths are unique
Uniqueness of strengths

Every morphism \( f : \Gamma \triangleright_{D} X \to Y \) can be applied at points of \( \Gamma \):

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\langle f \rangle : V(I, \Gamma) \to D(X, Y)
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y \mapsto \left( X \xrightarrow{\lambda^{-1}} I \triangleright_{D} X \xrightarrow{\gamma \triangleright_{D} X} \Gamma \triangleright_{D} X \xrightarrow{f} Y \right)
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Every strength for \( F : C \to D \) satisfies

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\]

so if \( \triangleright_{D} \) is well-pointed (\( \langle \_ \rangle \) is injective), strengths are unique

Example: every \( F : \textbf{Set} \to \textbf{Set} \) has a unique strength

\[
\text{str}_{\Gamma, X}(\gamma, t) = F(x \mapsto (\gamma, x)) t
\]
Uniqueness of strengths

Every morphism $f : \Gamma \triangleright_D X \to Y$ can be applied at points of $\Gamma$:

$$ \langle f \rangle : V(I, \Gamma) \to D(X, Y) $$

$$ \gamma \mapsto \left( X \xrightarrow{\lambda^{-1}} I \triangleright_D X \xrightarrow{\gamma \triangleright_D X} \Gamma \triangleright_D X \xrightarrow{f} Y \right) $$

Every strength for $F : C \to D$ satisfies

$$ \langle \text{str}_{\Gamma, X} \rangle \gamma = F((\gamma \triangleright_C X) \circ \lambda^{-1}) $$

so if $\triangleright_D$ is well-pointed ($\langle \langle - \rangle \rangle$ is injective), strengths are unique

Example: $\langle \langle - \rangle \rangle : (X, \leq) \leftrightarrow (X, =) : \text{Poset} \to \text{Poset}$ has no strength:

$$ \text{str}_{\Gamma, X} : (\gamma, x) \leftrightarrow (\gamma, x) : \Gamma \times |X| \to |\Gamma \times X| \text{ is not monotone} $$
Existence of strengths

If for every

$$\zeta : V(I, \Gamma) \to D(X, Y)$$

there is a

$$\Phi_{\Gamma}\zeta : \Gamma \triangleright_D X \to Y$$

satisfying $$(\Phi_{\Gamma}\zeta) = \zeta$$ and respecting the action structure of $$\triangleright_D$$, then

- every functor $$F : C \to D$$ forms a strong functor
- in a coherent way (natural transformations are strong)

Example: for $$(D, \triangleright) = (\text{Set}, \times)$$, take $$\Phi_{\Gamma}\zeta(y, x) = \zeta y x$$, then

$$F^{(\Gamma)} f (y, t) = F(x \mapsto f(y, x)) t$$
Existence of strengths

If for every
\[ \zeta : V(I, \Gamma) \rightarrow D(X, Y) \]
there is a
\[ \Phi_\Gamma \zeta : \Gamma \triangleright_D X \rightarrow Y \]
satisfying \( (\Phi_\Gamma \zeta) = \zeta \) and respecting the action structure of \( \triangleright_D \), then
\begin{itemize}
  \item every functor \( F : C \rightarrow D \) forms a strong functor
  \item in a coherent way (natural transformations are strong)
\end{itemize}

Example: for \( (D, \triangleright) = (\text{Set}_*, \times) \), take \( \Phi_\Gamma \zeta (y, x) = \zeta \star x \), then
\[ F^{(\Gamma)} f (y, t) = F(x \mapsto f(\star, x)) t \]
(since \( \text{Set}_*(1, \Gamma) = \{ \star \} \))
Strong monads

A strong monad on \((C, \triangleright_C)\) is:

- an object mapping \(X \mapsto TX\)
- with unit morphisms \(\eta_X : X \rightarrow TX\)
- and a Kleisli extension operation

\[
f : \Gamma \triangleright X \rightarrow TY
\]

\[
f^\dagger : \Gamma \triangleright TX \rightarrow TY
\]

- natural in \(\Gamma\) and satisfying three laws
Strong monads

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f : \Gamma \triangleright X \to TY \\
\implies f^\dagger : \Gamma \triangleright TX \to TY
\]

- natural in \(\Gamma\) and satisfying three laws

Example: take \((C, \triangleright) = (\text{Set}, \times)\) and

- \(\text{List}X = \text{lists over } X\)
- \(\eta x = [x]\)
- \(f^\dagger(y, [x_1, \ldots, x_n]) = f(y, x_1) + \cdots + f(y, x_n)\)
Strong monads

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- natural in \(\Gamma\) and satisfying three laws

Example: take \((C, \triangleright) = ([\mathbb{N}, \text{Set}], \times)\) and

- \(X \mapsto \mathbb{N} \times X\) (with \(e(n, x) = (n, e \cdot x)\))
- \(\eta x = (0, x)\)
- \(f^\dagger(y, (n, x)) = (n + m, y)\) where \((m, y) = f(y, x)\)
  - or-
  \(f^\dagger(y, (n, x)) = (n + m, y)\) where \((m, y) = f(e^n y, x)\)
Strong monads

A strong monad on \((C, \triangleright_C)\) is:

- an object mapping \(X \mapsto TX\)
- with unit morphisms \(\eta_X : X \to TX\)
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\begin{align*}
  f : \Gamma \triangleright X \to TY \\
  f^\dagger : \Gamma \triangleright TX \to TY
\end{align*}
\]

- natural in \(\Gamma\) and satisfying three laws

Equivalently:

- A strong functor \(T\), with a strong unit and a strong multiplication, satisfying the monad laws
- A monad \((T, \eta, \mu)\) with a lifting of \(\triangleright\) to \(\text{Kl} T\)

\[
\begin{align*}
  \xymatrix{
    V \times C \ar@<1.5ex>[r]^-{\triangleright} \ar[d]_{V \times K_T} & C \ar[d]^{K_T} \\
    V \times \text{Kl} T \ar[r]_-{\triangleright_T} & \text{Kl} T
  }
\end{align*}
\]
Uniqueness and existence of strengths

- If $\triangleright$ is well-pointed, then strengths are unique

$$[\text{str}_{\Gamma,X}] \gamma = T((\gamma \triangleright_c X) \circ \lambda^{-1})$$

- Existence doesn’t work as well: $\Phi$ makes $\text{Id}$ into a strong monad only if $\triangleright$ is well-pointed
  - On $\text{Set}_*$, defining $f^\dagger(\gamma, x) = f(\star, x)$ does not make $\text{Id}$ into a strong monad
Equivalent perspectives: enrichment

For each $C$, if $\vdash X \vdash X \rightharpoonup -$ : $C \rightarrow V$ for each $X$:

- to make $\rhd$ into an action is equivalently
- to make $\rightarrow$ into an enrichment of $C$ over $V$ such that
  
  $$(\Gamma \rhd X) \rightarrow Y \cong \Gamma \mapsto (X \rightarrow Y)$$

(when $V$ is left closed)
Equivalent perspectives: enrichment

For each $C$, if $\triangleright X \dashv X \rightarrow - : C \rightarrow V$ for each $X$:

- to make $\triangleright$ into an action is equivalently
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(when $V$ is left closed)

Under this bijection:

- strong functors are the same as enriched functors
  $$X \mapsto FX$$
  $$\text{fmap}_{X,Y} : (X \rightarrow Y) \rightarrow (FX \rightarrow FY)$$

- strong monads are the same as enriched monads
  $$X \mapsto TX$$
  $$\eta_X : X \rightarrow TX$$
  $$\text{bind}_{X,Y} : (X \rightarrow TY) \rightarrow (TX \rightarrow TY)$$
Equivalent perspectives: enrichment

For each $C$, if $\rightarrow X \vdash X \rightarrow - : C \rightarrow V$ for each $X$:

- to make $\rightarrow$ into an action is equivalently
- to make $\rightarrow$ into an enrichment of $C$ over $V$ such that
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(when $V$ is left closed)

Under this bijection:

- strong functors are the same as enriched functors
  $$X \rightarrow FX$$
  $$\text{class Functor } f \text{ where}$$
  $$\text{fmap} :: (a \rightarrow b) \rightarrow f a \rightarrow f b$$

- strong monads are the same as enriched monads
  $$X \rightarrow TX$$
  $$\text{class Monad } t \text{ where}$$
  $$\text{return} :: a \rightarrow t a$$
  $$(>>=) :: (a \rightarrow t b) \rightarrow t a \rightarrow t b$$
There are many different ways of formulating strength arising by looking at strength from different perspectives leading to various different properties (existence, uniqueness, etc.)

Some other things (in the paper):

- Third perspective: powering $\Gamma \triangleright \dashv \Gamma \triangleleft \dashv : \mathcal{C} \to \mathcal{C}$
  $\leadsto$ formulation of strength in terms of $\text{Alg}(T)$
  $\leadsto$ strengths for free monads
- Bistrengths and commutative monads