# Flexibly graded monads and graded algebras

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Algebraic structures appear in models of effects using monads

- especially for presentations and algebraic operations
- e.g. monoids and finite nondeterminism, mnemoids and global state

Is there are similar story about graded algebraic structures?

- e.g. grading nondeterminism by number of choices?

$$\frac{\Gamma \vdash M_1 : A \,\&\, d_1 \qquad \Gamma \vdash M_2 : A \,\&\, d_2}{\Gamma \vdash \texttt{or}(M_1, M_2) : A \,\&\, (d_1 + d_2)} \qquad \frac{}{\Gamma \vdash \texttt{fail}() : A \,\&\, 0}$$

$$(d, d', d_1, d_2 \in \mathbb{N})$$

$$\frac{\Gamma \vdash M : A \,\&\, d \qquad d \leq d'}{\Gamma \vdash M : A \,\&\, d'}$$

Motivation: develop a notion of presentation for graded monads (see our ICFP paper)

# Monoids and nondeterminism

If there is a monad T on $\mathbf{Set}$ whose algebras

$$A \in \mathbf{Set} \qquad a : TA \to A$$

are exactly monoids

$$A \in \mathbf{Set} \qquad u : \mathbb{1} \to A \qquad m : A \times A \to A$$

then from the free algebras

$$TX \in \mathbf{Set} \qquad \mu_X : T(TX) \to TX$$

we get algebraic operations [Plotkin and Power '02]

$$\mathsf{fail}_X : \mathbb{1} \to TX \qquad \mathsf{or}_X : TX \times TX \to TX$$

we can use to model finite nondeterminism

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T is the standard list monad List:

$$\mathsf{List}\,X = \text{lists over } X \qquad \mathsf{fail}_X\,() = [] \qquad \mathsf{or}_X(\mathrm{xs}_1, \mathrm{xs}_2) = \mathrm{xs}_1 +\!\!+\, \mathrm{xs}_2$$

# Graded monoids and nondeterminism

If there is a graded monad T on **Set** whose algebras

$$A : \mathbb{N}_{\leq} \to \mathbf{Set} \qquad a_{d,e} : T(Ae)d \to A(d \cdot e)$$

are exactly graded monoids

$$A : \mathbb{N}_{\leq} \to \mathbf{Set} \qquad u : \mathbb{1} \to A0 \qquad m_{d_1,d_2} : Ad_1 \times Ad_2 \to A(d_1 + d_2)$$

then from the free algebras

$$TX : \mathbb{N}_{\leq} \to \mathbf{Set} \qquad \mu_{X,d,e} : T(TXe)d \to TX(d \cdot e)$$

we get algebraic operations

$$\mathsf{fail}_X : \mathbb{1} \to TX0 \qquad \mathsf{or}_{X,d_1,d_2} : TXd_1 \times TXd_2 \to TX(d_1 + d_2)$$

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we can use to model finite nondeterminism

But there is no such graded monad

# Graded monads

A ($\mathbb{N}_{\leq}$–)graded set $A$ is a functor $\mathbb{N}_{\leq} \to \mathbf{Set}$:

- a set $Ad$ for each $d \in \mathbb{N}$
- with a function $Ad \to Ad'$ for each $d \leq d'$, respecting reflexivity, transitivity of $\leq$

A graded monad T consists of: [Smirnov '08, Melliès '12, Katsumata '14]

- a graded set $TX$ for each (ungraded) set $X$
- unit functions $\eta_X : X \to TX1$
- Kleisli extension $\dfrac{f : X \to TYe}{f^\dagger_d : TXd \to TY(d \cdot e)}$ $(d, e \in \mathbb{N})$

satisfying (graded) monad laws

For example: List$Xe$ = lists over $X$ of length $\leq e$, with

$$\eta_X x = [x] \qquad f^\dagger_d[x_1, \dots, x_k] = f x_1 +\!\!+ \cdots +\!\!+ f x_k$$

# Graded monoids and nondeterminism

The algebras of

$$\mathsf{List}\,X e = \text{lists over } X \text{ of length } \leq e$$

have the form

$$A : \mathbb{N}_{\leq} \to \mathbf{Set} \qquad a_{d,e} : \mathsf{List}(Ae)d \to A(d \cdot e)$$

These do not form graded monoids

$$u : \mathbb{1} \to A0 \qquad m_{d_1,d_2} : Ad_1 \times Ad_2 \to A(d_1 + d_2)$$

But the free algebras $\mathsf{List}\,X : \mathbb{N}_{\leq} \to \mathbf{Set}$ do:

$$\begin{array}{ll} \mathsf{fail}_X : \mathbb{1} \to \mathsf{List}\,X0 & \mathsf{or}_{X,d_1,d_2} : \mathsf{List}\,Xd_1 \times \mathsf{List}\,Xd_2 \to \mathsf{List}\,X(d_1 + d_2) \\ \mathsf{fail}_X() = [] & \mathsf{or}_{X,d_1,d_2}(\mathrm{xs}_1, \mathrm{xs}_2) = \mathrm{xs}_1 \mathbin{+\!\!+} \mathrm{xs}_2 \end{array}$$

# Graded algebraic structures

Graded sets $A$:

- a set $Ad$ for each $d \in \mathbb{N}$
- with a function $(d{\le}d')^* : Ad \to Ad'$ for each $d \le d'$
- such that $\mathrm{id}_{Ad} = (d{\le}d)^*$ and $(d'{\le}d'')^* \circ (d{\le}d')^* = (d{\le}d'')^*$

Morphisms $f : A \to B$:

$$f_d : Ad \to Bd \quad \text{for each } d \in \mathbb{N} \qquad \text{natural in } d$$

Graded monoids $\mathsf{A} = (A, u, m)$:

- A graded set $A$
- with functions $u : \mathbb{1} \to A0$ and $m_{d_1,d_2} : Ad_1 \times Ad_2 \to A(d_1 + d_2)$
- natural in $d_1, d_2$, and satisfying unitality and associativity laws

Morphisms $f : \mathsf{A} \to \mathsf{B}$:

$$f : A \to B \qquad \text{such that } f_0(u()) = u() \text{ and } f_{d_1+d_2}(m_{d_1,d_2}(x, y)) = m_{d_1,d_2}(f_{d_1}x, f_{d_2}y)$$

Forgetful functor $U : \mathbf{GMon} \to \mathbf{GSet}$

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Morphisms $f : A \mathrel{-e\!\rightarrow} B$ of grade $e$:

$$f_d : Ad \to B(d \cdot e) \quad \text{for each } d \in \mathbb{N} \qquad \text{natural in } d$$

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Forgetful functor $U : \mathbf{GMon} \to \mathbf{GSet}$

# Locally graded categories [Wood '76]

A *locally graded category* $\mathcal{C}$ has:

- a collection $|\mathcal{C}|$ of objects
- graded sets $\mathcal{C}(X, Y)$ of morphisms $\quad$ ($f : X \xrightarrow{e} Y$ means $f \in \mathcal{C}(X, Y)e$)
- identities $\mathrm{id}_X : X \xrightarrow{1} X$
- composition

$$\frac{f : X \xrightarrow{e} Y \qquad g : Y \xrightarrow{e'} Z}{g \circ f : X \xrightarrow{e \cdot e'} Z}$$

  natural in $e, e'$

such that

$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

(These are categories enriched over $[\mathbb{N}_{\leq}, \mathbf{Set}]$ with Day convolution)

Graded algebraic structures form locally graded categories that forget into $\mathbf{GSet}$

$$U : \mathbf{GMon} \to \mathbf{GSet}$$

# Relative monads [Altenkirch, Chapman, Uustalu '15]

## Definition

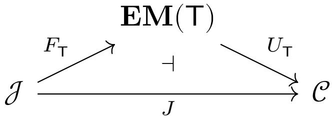
A *$J$-relative monad* T (for $J : \mathcal{J} \to \mathcal{C}$) consists of:

- object mapping $T : |\mathcal{J}| \to |\mathcal{C}|$
- unit morphisms $\eta_X : JX \xrightarrow{1} TX$
- Kleisli extension $\dfrac{f : JX \xrightarrow{e} TY}{f^\dagger : TX \xrightarrow{e} TY}$ natural in $e$

such that the monad laws hold:

$$f^\dagger \circ \eta_X = f \qquad \eta_X^\dagger = \mathrm{id}_{TX} \qquad (g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger$$

T has an Eilenberg-Moore resolution



# Graded monads

Ungraded sets form a full locally graded subcategory of graded sets:

$$K : \mathbf{RSet} \hookrightarrow \mathbf{GSet}$$

$$KXd = \begin{cases} X & \text{if } d \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

($\mathbf{RSet}$ is the free locally graded category on $\mathbf{Set}$)

so that $\mathbf{GSet}(KX, Y)e \cong \mathbf{Set}(X, Ye)$, and

graded monads are $K$-relative monads

graded monad:

- object mapping $T : |\mathbf{Set}| \to |\mathbf{GSet}|$
- unit functions $\eta_X : X \to TX1$
- Kleisli extension $\dfrac{f : X \to TYe}{f_d^\dagger : TXd \to TY(d \cdot e)}$

$K$-relative monad:

- object mapping $T : |\mathbf{Set}| \to |\mathbf{GSet}|$
- unit morphisms $\eta_X : KX \xrightarrow{1} TX$
- Kleisli extension $\dfrac{f : KX \xrightarrow{e} TY}{f^\dagger : TX \xrightarrow{e} TY}$

# Graded monoids

There is no graded monad T such that

$$\begin{array}{ccc} \mathbf{GMon} & \xrightarrow{\cong} & \mathbf{EM}(\mathsf{T}) \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle U_{\mathsf{T}}} & \\ & \mathbf{GSet} & \end{array}$$

But we do have:

$$\begin{array}{ccc} \mathbf{GMon} & \xrightarrow{R} & \mathbf{EM}(\mathsf{List}) \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle U_{\mathsf{List}}} & \\ & \mathbf{GSet} & \end{array}$$

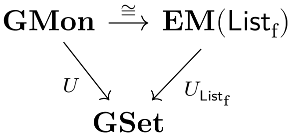
# Flexibly graded monads

(Locally graded) monads T on $\mathbf{GSet}$ (i.e. $\mathrm{Id}_{\mathbf{GSet}}$-relative monads):

- a graded set $TX$ for each graded set $X$
- unit $\eta : X \mathrel{-1\!\!\to} TX$
- Kleisli extension $\dfrac{f : X \mathrel{-e\!\!\to} TY}{f^\dagger : TX \mathrel{-e\!\!\to} TY}$ (or multiplication $\mu_X : T(TX) \mathrel{-1\!\!\to} TX$)
- satisfying monad laws

Example: lists

$\mathsf{List_f} X d =$ "lists over $X$, with total grade at most $d$"




Formally $\mathsf{List_f} X d = \mathrm{colim}_{\vec{d}' \in S_d} \prod_i X d'_i$
where $S_d$ is the poset of lists $(d'_1, \ldots, d'_n)$ such that $d \geq \sum_i d'_i$,
ordered pointwise

# Constructing graded monads

Every flexibly graded monad $\mathsf{T}$ restricts to a graded monad $\lfloor \mathsf{T} \rfloor$ by

$$\lfloor T \rfloor X = T(KX)$$

and this comes with $R_{\mathsf{T}} : \mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\lfloor \mathsf{T} \rfloor)$, commuting with forgetful functors

The restriction is universal:

$$\begin{array}{ccc} \mathbf{EM}(\mathsf{T}) & \xrightarrow{R_{\mathsf{T}}} & \mathbf{EM}(\lfloor \mathsf{T} \rfloor) \\ & {\scriptstyle R'} \searrow & \downarrow {\scriptstyle \mathbf{EM}(\alpha)} \\ & & \mathbf{EM}(\mathsf{T}') \end{array} \qquad \begin{array}{c} \lfloor \mathsf{T} \rfloor \\ {\scriptstyle \alpha} \uparrow \\ \mathsf{T}' \end{array}$$

(where $R_{\mathsf{T}}, R'$ commute with forgetful functors)

And free $\lfloor \mathsf{T} \rfloor$-algebras are free $\mathsf{T}$-algebras:

$$\begin{array}{ccc} \mathbf{RSet} & \xrightarrow{F_{\lfloor \mathsf{T} \rfloor}} & \mathbf{EM}(\lfloor \mathsf{T} \rfloor) \\ {\scriptstyle K} \downarrow & & \uparrow {\scriptstyle R_{\mathsf{T}}} \\ \mathbf{GSet} & \xrightarrow[F_{\mathsf{T}}]{} & \mathbf{EM}(\mathsf{T}) \end{array}$$

# Constructing graded monads

Since $\lfloor \mathsf{List}_{\mathrm{f}} \rfloor \cong \mathsf{List}$:

$$\begin{array}{ccc} \mathbf{GMon} & \xrightarrow{R} & \mathbf{EM}(\mathsf{List}) \\ & {\scriptstyle R'}\searrow & \downarrow{\scriptstyle \mathbf{EM}(\alpha)} \\ & & \mathbf{EM}(\mathsf{T}') \end{array} \qquad \begin{array}{c} \mathsf{List} \\ {\scriptstyle \alpha}\uparrow \\ \mathsf{T}' \end{array}$$

for every graded monad $\mathsf{T}'$ and $R' : \mathbf{GMon} \to \mathsf{T}'$ commuting with forgetful functors

So:

- no graded monad has graded monoids as algebras
- but List is the best we can do
- and free List-algebras form free graded monoids

# Conclusions

Graded monads

- don't exactly capture certain algebraic structures
- but do get close enough to model effects in a canonical way

Locally graded categories are a good setting for doing grading:

- algebraic structures form locally graded categories
- these structures are often captured by flexibly graded monads (i.e. enriched monads on $\mathbf{GSet}$)
- graded monads are relative monads