Flexibly graded monads and graded algebras

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Algebraic structures appear in models of effects using monads

- especially for presentations and algebraic operations
- e.g. monoids and finite nondeterminism, mnemoids and global state

Is there are similar story about graded algebraic structures?

- e.g. grading nondeterminism by number of choices?

$$\frac{\Gamma \vdash M_1 : A \mathbin{\&} d_1 \qquad \Gamma \vdash M_2 : A \mathbin{\&} d_2}{\Gamma \vdash \mathtt{or}(M_1, M_2) : A \mathbin{\&} (d_1 + d_2)} \qquad \frac{}{\Gamma \vdash \mathtt{fail}() : A \mathbin{\&} 0}$$

$$\frac{\Gamma \vdash M : A \mathbin{\&} d \qquad d \leq d'}{\Gamma \vdash M : A \mathbin{\&} d'} \qquad (d, d', d_1, d_2 \in \mathbb{N})$$

Motivation: develop a notion of presentation for graded monads (see our ICFP paper)

Monoids and nondeterminism

If there is a monad T on $\mathbf{Set}$ whose algebras

$$A \in \mathbf{Set} \qquad a : TA \to A$$

are exactly monoids

$$A \in \mathbf{Set} \qquad u : 1 \to A \qquad m : A \times A \to A$$

then from the free algebras

$$TX \in \mathbf{Set} \qquad \mu_X : T(TX) \to TX$$

we get algebraic operations [Plotkin and Power '02]

$$\mathsf{fail}_X : 1 \to TX \qquad \mathsf{or}_X : TX \times TX \to TX$$

we can use to model finite nondeterminism

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T is the standard list monad List:

$$\mathsf{List}\, X = \text{lists over } X \qquad \mathsf{fail}_X\,() = [] \qquad \mathsf{or}_X(\mathrm{xs}_1, \mathrm{xs}_2) = \mathrm{xs}_1 +\!\!+\, \mathrm{xs}_2$$

Graded monoids and nondeterminism

If there is a graded monad T on $\mathbf{Set}$ whose algebras

$$A : \mathbb{N}_{\leq} \to \mathbf{Set} \qquad a_{d,e} : T(Ae)d \to A(d \cdot e)$$

are exactly graded monoids

$$A : \mathbb{N}_{\leq} \to \mathbf{Set} \qquad u : \mathbb{1} \to A0 \qquad m_{d_1,d_2} : Ad_1 \times Ad_2 \to A(d_1 + d_2)$$

then from the free algebras

$$TX : \mathbb{N}_{\leq} \to \mathbf{Set} \qquad \mu_{X,d,e} : T(TXe)d \to TX(d \cdot e)$$

we get algebraic operations

$$\mathsf{fail}_X : \mathbb{1} \to TX0 \qquad \mathsf{or}_{X,d_1,d_2} : TXd_1 \times TXd_2 \to TX(d_1 + d_2)$$

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But there is no such graded monad

Graded monads

A ($\mathbb{N}_{\leq}$–)graded set A is a functor $\mathbb{N}_{\leq} \to \mathbf{Set}$:

- a set Ad for each $d \in \mathbb{N}$
- with a function $Ad \to Ad'$ for each $d \leq d'$, respecting reflexivity, transitivity of $\leq$

A graded monad T consists of: [Smirnov '08, Melliès '12, Katsumata '14]

- a graded set TX for each (ungraded) set X
- unit functions $\eta_X : X \to TX1$
- Kleisli extension $\dfrac{f : X \to TYe}{f_d^{\dagger} : TXd \to TY(d \cdot e)}$ $(d, e \in \mathbb{N})$

satisfying (graded) monad laws

For example: ListXe = lists over X of length $\leq e$, with

$$\eta_X x = [x] \qquad f_d^{\dagger}[x_1, \ldots, x_k] = f x_1 +\!\!+ \cdots +\!\!+ f x_k$$

Graded monoids and nondeterminism

The algebras of

$$\mathsf{List}Xe = \text{lists over } X \text{ of length } \leq e$$

have the form

$$A : \mathbb{N}_{\leq} \to \mathbf{Set} \qquad a_{d,e} : \mathsf{List}(Ae)d \to A(d \cdot e)$$

These do not form graded monoids

$$u : \mathbb{1} \to A0 \qquad m_{d_1,d_2} : Ad_1 \times Ad_2 \to A(d_1 + d_2)$$

But the free algebras $\mathsf{List}X : \mathbb{N}_{\leq} \to \mathbf{Set}$ do:

$$\begin{array}{ll} \mathsf{fail}_X : \mathbb{1} \to \mathsf{List}X0 & \mathsf{or}_{X,d_1,d_2} : \mathsf{List}Xd_1 \times \mathsf{List}Xd_2 \to \mathsf{List}X(d_1 + d_2) \\ \mathsf{fail}_X() = [] & \mathsf{or}_{X,d_1,d_2}(\mathrm{xs}_1, \mathrm{xs}_2) = \mathrm{xs}_1 \mathbin{+\!\!+} \mathrm{xs}_2 \end{array}$$

Graded algebraic structures

Graded sets A:

- a set Ad for each $d \in \mathbb{N}$
- with a function $(d{\le}d')^* : Ad \to Ad'$ for each $d \le d'$
- such that $\mathrm{id}_{Ad} = (d{\le}d)^*$ and $(d'{\le}d'')^* \circ (d{\le}d')^* = (d{\le}d'')^*$

Morphisms $f : A \to B$:

$$f_d : Ad \to Bd \quad \text{for each } d \in \mathbb{N} \qquad\qquad \text{natural in } d$$

Graded monoids $\mathsf{A} = (A, u, m)$:

- A graded set A
- with functions $u : \mathbb{1} \to A0$ and $m_{d_1,d_2} : Ad_1 \times Ad_2 \to A(d_1 + d_2)$
- natural in d_1, d_2, and satisfying unitality and associativity laws

Morphisms $f : \mathsf{A} \to \mathsf{B}$:

$$f : A \to B \qquad \text{such that } f_0(u()) = u() \text{ and } f_{d_1+d_2}(m_{d_1,d_2}(x, y)) = m_{d_1,d_2}(f_{d_1}x, f_{d_2}y)$$

Forgetful functor $U : \mathbf{GMon} \to \mathbf{GSet}$

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Morphisms $f : A \xrightarrow{e} B$ of grade e:

$$f_d : Ad \to B(d \cdot e) \quad \text{for each } d \in \mathbb{N} \qquad \text{natural in } d$$

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Forgetful functor $U : \mathbf{GMon} \to \mathbf{GSet}$

Locally graded categories [Wood '76]

A *locally graded category* $\mathcal{C}$ has:

- a collection $|\mathcal{C}|$ of objects
- graded sets $\mathcal{C}(X, Y)$ of morphisms $\quad$ ($f : X \,{-e\!\to}\, Y$ means $f \in \mathcal{C}(X, Y)e$)
- identities $\mathrm{id}_X : X \,{-1\!\to}\, X$
- composition

$$\frac{f : X \,{-e\!\to}\, Y \qquad g : Y \,{-e'\!\to}\, Z}{g \circ f : X \,{-e \cdot e'\!\to}\, Z}$$

  natural in e, e'

such that

$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

(These are categories enriched over $[\mathbb{N}_{\leq}, \mathbf{Set}]$ with Day convolution)

Graded algebraic structures form locally graded categories that forget into $\mathbf{GSet}$

$$U : \mathbf{GMon} \to \mathbf{GSet}$$

Relative monads [Altenkirch, Chapman, Uustalu '15]

Definition

A *J-relative monad* T (for $J : \mathscr{J} \to \mathscr{C}$) consists of:

- object mapping $T : |\mathscr{J}| \to |\mathscr{C}|$
- unit morphisms $\eta_X : JX \xrightarrow{1} TX$
- Kleisli extension $\dfrac{f : JX \xrightarrow{e} TY}{f^\dagger : TX \xrightarrow{e} TY}$ natural in e

such that the monad laws hold:

$$f^\dagger \circ \eta_X = f \qquad \eta_X^\dagger = \mathrm{id}_{TX} \qquad (g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger$$

T has an Eilenberg-Moore resolution

$$\begin{tikzcd} & \mathbf{EM}(\mathsf{T}) \arrow[dr, "U_{\mathsf{T}}"] \arrow[d, phantom, "\dashv"] & \\ \mathscr{J} \arrow[ur, "F_{\mathsf{T}}"] \arrow[rr, "J"'] & {} & \mathscr{C} \end{tikzcd}$$

Graded monads

Ungraded sets form a full locally graded subcategory of graded sets:

$$K : \mathbf{RSet} \hookrightarrow \mathbf{GSet}$$

$$KXd = \begin{cases} X & \text{if } d \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

($\mathbf{RSet}$ is the free locally graded category on $\mathbf{Set}$)

so that $\mathbf{GSet}(KX, Y)e \cong \mathbf{Set}(X, Ye)$, and

graded monads are K-relative monads

graded monad:

- object mapping $T : |\mathbf{Set}| \to |\mathbf{GSet}|$
- unit functions $\eta_X : X \to TX1$
- Kleisli extension $\dfrac{f : X \to TYe}{f^\dagger_d : TXd \to TY(d \cdot e)}$

K-relative monad:

- object mapping $T : |\mathbf{Set}| \to |\mathbf{GSet}|$
- unit morphisms $\eta_X : KX \xrightarrow{1} TX$
- Kleisli extension $\dfrac{f : KX \xrightarrow{e} TY}{f^\dagger : TX \xrightarrow{e} TY}$

Graded monoids

There is no graded monad T such that

$$\begin{tikzcd} \mathbf{GMon} \arrow[rr, "\cong"] \arrow[dr, "U"'] & & \mathbf{EM}(\mathsf{T}) \arrow[dl, "U_{\mathsf{T}}"] \\ & \mathbf{GSet} & \end{tikzcd}$$

But we do have:

$$\begin{tikzcd} \mathbf{GMon} \arrow[rr, "R"] \arrow[dr, "U"'] & & \mathbf{EM}(\mathsf{List}) \arrow[dl, "U_{\mathsf{List}}"] \\ & \mathbf{GSet} & \end{tikzcd}$$

Flexibly graded monads

(Locally graded) monads T on $\mathbf{GSet}$ (i.e. $\mathrm{Id}_{\mathbf{GSet}}$-relative monads):

- a graded set TX for each graded set X
- unit $\eta : X \mathrel{-1\!\to} TX$
- Kleisli extension $\dfrac{f : X \mathrel{-e\!\to} TY}{f^{\dagger} : TX \mathrel{-e\!\to} TY}$ (or multiplication $\mu_X : T(TX) \mathrel{-1\!\to} TX$)
- satisfying monad laws

Example: lists

$\mathsf{List}_{\mathrm{f}} X d =$ "lists over X, with total grade at most d"

$$\begin{tikzcd} \mathbf{GMon} \arrow[rr, "\cong"] \arrow[dr, "U"'] & & \mathbf{EM}(\mathsf{List}_{\mathrm{f}}) \arrow[dl, "U_{\mathsf{List}_{\mathrm{f}}}"] \\ & \mathbf{GSet} & \end{tikzcd}$$

Formally $\mathsf{List}_{\mathrm{f}} X d = \mathrm{colim}_{\vec{d}' \in S_d} \prod_i X d'_i$
where S_d is the poset of lists $(d'_1, \ldots, d'_n)$ such that $d \geq \sum_i d'_i$,
ordered pointwise

Constructing graded monads

Every flexibly graded monad T restricts to a graded monad $\lfloor \mathsf{T} \rfloor$ by

$$\lfloor T \rfloor X = T(KX)$$

and this comes with $R_{\mathsf{T}} : \mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\lfloor \mathsf{T} \rfloor)$, commuting with forgetful functors

The restriction is universal:

$$\begin{tikzcd} \mathbf{EM}(\mathsf{T}) \arrow[r, "R_{\mathsf{T}}"] \arrow[dr, "R'"'] & \mathbf{EM}(\lfloor \mathsf{T} \rfloor) \arrow[d, "\mathbf{EM}(\alpha)"] \\ & \mathbf{EM}(\mathsf{T}') \end{tikzcd} \qquad \begin{tikzcd} \lfloor \mathsf{T} \rfloor \\ \mathsf{T}' \arrow[u, dashed, "\alpha"] \end{tikzcd}$$

(where R_{T}, R' commute with forgetful functors)

And free $\lfloor \mathsf{T} \rfloor$-algebras are free T-algebras:

$$\begin{tikzcd} \mathbf{RSet} \arrow[r, "F_{\lfloor \mathsf{T} \rfloor}"] \arrow[d, "K"'] & \mathbf{EM}(\lfloor \mathsf{T} \rfloor) \\ \mathbf{GSet} \arrow[r, "F_{\mathsf{T}}"'] & \mathbf{EM}(\mathsf{T}) \arrow[u, "R_{\mathsf{T}}"'] \end{tikzcd}$$

Constructing graded monads

Since $\lfloor \mathsf{List}_{\mathrm{f}} \rfloor \cong \mathsf{List}$:

$$\begin{tikzcd} \mathbf{GMon} \arrow[r, "R"] \arrow[dr, "R'"'] & \mathbf{EM}(\mathsf{List}) \arrow[d, "\mathbf{EM}(\alpha)"] & \mathsf{List} \\ & \mathbf{EM}(\mathsf{T}') & \mathsf{T}' \arrow[u, dashed, "\alpha"] \end{tikzcd}$$

for every graded monad T' and $R' : \mathbf{GMon} \to \mathsf{T}'$ commuting with forgetful functors

So:

- no graded monad has graded monoids as algebras
- but List is the best we can do
- and free List-algebras form free graded monoids

Conclusions

Graded monads

- don't exactly capture certain algebraic structures
- but do get close enough to model effects in a canonical way

Locally graded categories are a good setting for doing grading:

- algebraic structures form locally graded categories
- these structures are often captured by flexibly graded monads (i.e. enriched monads on $\mathbf{GSet}$)
- graded monads are relative monads

See our MPC 2022 paper:

`https://dylanm.org/drafts/flexibly-graded-monads.pdf`