Canonical gradings of monads

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The writer monad $\text{Wr}$ for lists over a set $C$ has:

- **Object mapping** $\text{Wr} : \text{Set} \rightarrow \text{Set}$
  
  $\text{Wr} X = \text{List } C \times X$

- **Unit functions** $\eta_X : X \rightarrow \text{Wr } X$
  
  $\eta_X x = ([], x)$

- **Multiplication functions** $\mu_X : \text{Wr (Wr } X) \rightarrow \text{Wr } X$
  
  $\mu_X (s_1, (s_2, x)) = (s_1 + s_2, x)$
Example

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- unit functions $\eta_X : X \to \text{Wr } X$; $\eta_X x = ([], x)$

- multiplication functions $\mu_X : \text{Wr } (\text{Wr } X) \to \text{Wr } X$; $\mu_X (s_1, (s_2, x)) = (s_1 \oplus s_2, x)$

We can grade this by

- natural numbers $e \in \mathbb{N}$, to get a graded monad $\text{WrL}$:

$\text{WrL } e X = \text{List}_{\leq e} C \times X$; $\eta : X \to \text{WrL } 0 X$; $\mu : \text{WrL } e_1 (\text{WrL } e_2 X) \to \text{WrL } (e_1 + e_2) X$

- subsets $e \subseteq C$, to get a graded monad $\text{WrS}$:

$\text{WrS } e X = \text{List } e \times X$; $\eta : X \to \text{WrS } 0 X$; $\mu : \text{WrS } e_1 (\text{WrS } e_2 X) \to \text{WrS } (e_1 \cup e_2) X$
Example

The writer monad \( \text{Wr} \) for lists over a set \( C \) has:

- **object mapping** \( \text{Wr} : \text{Set} \to \text{Set} \)
  \[ \text{Wr} X = \text{List} C \times X \]

- **unit functions** \( \eta_X : X \to \text{Wr} X \)
  \[ \eta_X x = ([], x) \]

- **multiplication functions** \( \mu_X : \text{Wr} (\text{Wr} X) \to \text{Wr} X \)
  \[ \mu_X (s_1, (s_2, x)) = (s_1 + s_2, x) \]

We can grade this by

- **subsets** \( \Sigma \subseteq \text{List} C \), to get a graded monad \( \text{Wr}_C \):

  \[
  \text{Wr}_C \Sigma X = \Sigma \times X \quad \eta : X \to \text{Wr}_C J X \quad \mu : \text{Wr}_C \Sigma_1 (\text{Wr}_C \Sigma_2 X) \to \text{Wr}_C (\Sigma_1 \sqcup \Sigma_2) X
  \]

  where
  \[ J = \{[\[]\} \quad \Sigma_1 \sqcup \Sigma_2 = \{s_1 + s_2 \mid s_1 \in \Sigma_1, s_2 \in \Sigma_2\} \]
WrC is the canonical grading of Wr:

▶ WrL is

\[ N \xrightarrow{F} \mathcal{P}(\text{List } C) \xrightarrow{\text{WrC}} [\text{Set}, \text{Set}] \]

where

\[ Fe = \text{List}_{\leq e} C \subseteq \text{List } C \]

▶ WrS is

\[ \mathcal{P} C \xrightarrow{F} \mathcal{P}(\text{List } C) \xrightarrow{\text{WrC}} [\text{Set}, \text{Set}] \]

where

\[ Fe = \text{List } e \subseteq \text{List } C \]
For every monad $T$ on $\text{Set}$:

- there is a notion of grading of $T$
- $T$ has a canonical grading
- every other grading factors through the canonical one

In particular, monads can be canonically graded by shapes
More generally, given a suitable notion of $\mathcal{M}$-subfunctor, for every monad $T$: 

- there is a notion of $\mathcal{M}$-grading of $T$
- $T$ has a canonical $\mathcal{M}$-grading
- every other $\mathcal{M}$-grading factors through the canonical one

In particular, monads can be canonically graded by shapes
Gradings

Let \( T \) be a monad on \( \text{Set} \):

\[
T : \text{Set} \to \text{Set} \quad \quad \eta : X \to TX \quad \mu : T(TX) \to TX
\]

A grading \( (E, G) \) of \( T \) consists of:

- an partially ordered monoid \( (E, \le, I, \odot) \) of grades \( e \in E \)
- a subset \( GeX \subseteq TX \) for each \( e \in E \) and set \( X \) such that
  - \( GeX \subseteq Ge'X \) for all \( e \le e' \) and \( X \)
  - \( G \) is closed under the monad structure of \( T \):
    - \( Tf : TX \to TY \) restricts to \( Gef : GeX \to GeY \) for each \( e \in E \) and \( f : X \to Y \)
    - \( \eta : X \to TX \) restricts to \( \eta : X \to GIX \) for each \( X \)
    - \( \mu : T(TX) \to TX \) restricts to \( \mu : Ge_1(Ge_2X) \to G(e_1 \odot e_2)X \) for each \( e_1, e_2 \in E \) and \( X \)

Fact: \( (G, \eta, \mu) \) is a graded monad
Gradings

Let $T$ be a monad on $\text{Set}$

$$T: \text{Set} \to \text{Set} \quad \eta: X \to TX \quad \mu: T(TX) \to TX$$

A grading $(E, G)$ of $T$ consists of:

- an partially ordered monoid $(E, \leq, I, \odot)$ of grades $e \in E$
- a subset $G_e X \subseteq TX$ for each $e \in E$ and set $X$ such that some conditions hold.

- $(E, \leq, I, \odot) = (\mathbb{N}, \leq, 0, +)$, and

$$\text{WrL } e X = \text{List}_{\leq e} C \times X \subseteq \text{List } C \times X = \text{Wr } X$$

- $(E, \leq, I, \odot) = (\mathcal{P} C, \subseteq, \emptyset, \cup)$, and

$$\text{WrS } e X = \text{List } e \times X \subseteq \text{List } C \times X = \text{Wr } X$$

- $(E, \leq, I, \odot) = (\mathcal{P}(\text{List} C), \subseteq, \emptyset, \cup)$, and

$$\text{WrS } e X = \text{List } e \times X \subseteq \text{List } C \times X = \text{Wr } X$$
Aside: constructing the ordered monoid

Given

- a set $E$
- a subfunctor $Ge \subseteq T$ for each $e \in E$

under some conditions we get a grading of $T$ by defining

\[
\begin{align*}
e &\leq e' \iff Ge \subseteq Ge' \\
I &= \text{smallest } e \text{ such that } \eta \text{ restricts to } \eta : X \to GeX \\
e_1 \odot e_2 &= \text{smallest } e \text{ such that } \mu \text{ restricts to } \mu : Ge_1(Ge_2X) \to GeX
\end{align*}
\]
Canonical grading of a functor

A grading \((\mathcal{E}, G)\) of a functor \(T\) consists of:

- a partially ordered set \((\mathcal{E}, \leq)\) of grades \(e \in \mathcal{E}\)
- a subfunctor \(G_e \subseteq T\) for each \(e \in \mathcal{E}\)

such that \(G_e \subseteq G_{e'}\) for all \(e \leq e'\)

The canonical grading \((\text{Sub}(T), \hat{T})\) of \(T\) has:

- poset \(\text{Sub}(T)\) (subfunctors of \(T\), ordered by pointwise inclusion)
- subfunctors \(\hat{T}S = S\)

Universal property:

for every other grading \((\mathcal{E}, G)\) of \(T\), there is a unique monotone \(F : \mathcal{E} \to \text{Sub}(T)\) such that

\(G_e = \hat{T}(Fe)\) for all \(e\)
The canonical grading \((\text{Sub}(T), \hat{T})\) of a monad \(T\) has

ordered monoid \((\text{Sub}(T), J, \sqsubseteq)\) where

\[
\text{J}X = \{\eta x \mid x \in X\}
\]

subfunctors \(\hat{T}S = S\) where

\[
(S_1 \circ S_2)X = \{\mu t \mid t \text{ is in the image of } S_1(S_2X) \hookrightarrow T(TX)\}
\]

Universal property:

for every other grading \((\mathcal{E}, G)\) of \(T\), there is a unique \(F : \mathcal{E} \rightarrow \text{Sub}(T)\) that is lax monoidal and satisfies \(Ge = \hat{T}(Fe)\) for all \(e\)

\[
e \leq e' \Rightarrow Fe \sqsubseteq Fe'
\]

\[
J \subseteq FI
\]

\[
Fe \circ Fe' \subseteq F(e \circ e')
\]
Example: writer

Take the writer monad \( \text{Wr} \)

\[
\text{Wr} X = \text{List} C \times X
\]

Subfunctors \( S \subseteq \text{Wr} \) are equivalently subsets

\[
\Sigma \subseteq \text{List} C
\]

via

\[
\Sigma = \{ s \in \text{List} C \mid (s, \star) \in S1 \}
\]

\[
S X = \{ (s, x) \in \text{List} C \times X \mid s \in \Sigma \}
\]

So the canonical grading is \( \mathcal{P}(\text{List} C) \) with

\[
\text{Wr}_C \Sigma X = \Sigma \times X
\]

\[
\Sigma \leq \Sigma' \iff \Sigma \subseteq \Sigma' \quad J = \{ [] \}
\]

\[
\Sigma_1 \boxplus \Sigma_2 = \{ s_1 \boxplus s_2 \mid s_1 \in \Sigma_1, s_2 \in \Sigma_2 \}
\]
Example: reader

Take the reader monad \( \text{Read}_V \) (for a set \( V \))

\[
\text{Read}_V X = V \Rightarrow X
\]

Subfunctors \( S \subseteq \text{Read}_V \) are equivalently upwards-closed sets

\[
\Sigma \subseteq \text{Equiv}_V
\]

of equivalence relations of \( V \), via

\[
\Sigma = \{ R \in \text{Equiv}_V \mid [-]_R \in S(V/R) \}
\]

\[
SX = \{ f : V \to X \mid \exists R \in \Sigma. \forall v,v'. v R v' \Rightarrow f v = f v' \}
\]

and these give a canonical grading \( (\text{Sub}(\text{Read}_V), \text{ReadC}_V) \)

Example: for \( FR = \{ R' \in \text{Equiv}_V \mid R \subseteq R' \} \), the graded monad

\[
\text{Equiv}_V \xrightarrow{F} \text{Sub}(\text{Read}_V) \xrightarrow{\text{ReadC}_V} [\text{Set}, \text{Set}]
\]

is

\[
\text{Read}'_V RX \cong (V/R) \Rightarrow X
\]
$\mathcal{M}$-gradings of functors

For a class $\mathcal{M}$ of natural transformations $\mapsto$, an $\mathcal{M}$-grading of a functor $T$ consists of

- a category $\mathcal{E}$
- a functor $G : \mathcal{E} \to [\text{Set}, \text{Set}]$
- a natural transformation $m_e : Ge \mapsto T$, whose components are in $\mathcal{M}$

The $\mathcal{M}$-subfunctors of $T$ form an $\mathcal{M}$-grading $(\mathcal{M}/T, \hat{T})$, with

$$\hat{T}(S, m) = S \xrightarrow{m} T$$

and this is canonical:

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{G} & [\text{Set}, \text{Set}]
\downarrow \cong
\mathcal{M}/T & \xrightarrow{\hat{T}} & \mathcal{M}/T
\end{array}$$

(more precisely, it is pseudoterminal in the 2-category of $\mathcal{M}$-gradings of $T$)

Example: $\mathcal{M} = \text{componentwise injective}$
$\mathcal{M}$-gradings of monads

For a class $\mathcal{M}$ of natural transformations $\rightarrowtail$, an $\mathcal{M}$-grading of a monad $T$ consists of

- a monoidal category $\mathcal{E}$
- a graded monad $G : \mathcal{E} \to [\text{Set}, \text{Set}]$
- a monoidal natural transformation $m_e : Ge \rightarrowtail T$, whose components are in $\mathcal{M}$

Under some conditions, the canonical $\mathcal{M}$-grading of the functor $T$ gives a canonical $\mathcal{M}$-grading of the monad $T$

$\mathcal{E}$
\[ \downarrow F \]
\[ \downarrow \cong \]
$\mathcal{M}/T \rightarrowtail$ $[\text{Set}, \text{Set}]$

(more precisely, it is pseudoterminal in the 2-category of $\mathcal{M}$-gradings of $T$)
\(\mathcal{M}\)-gradings of monads

If

- \(\mathcal{M}\) forms a factorization system \((\mathcal{E}, \mathcal{M})\) on \([\text{Set}, \text{Set}]\)

\[
\begin{array}{ccc}
F & \xrightarrow{f} & G \\
\downarrow{e} & & \downarrow{m} \\
S & \xrightarrow{} & (e \in \mathcal{E}, m \in \mathcal{M})
\end{array}
\]

- for each \(e : F \to F'\) in \(\mathcal{E}\) and \(G\),

\[
(e \cdot G) : F \cdot G \to F' \cdot G \quad (G \cdot e) : G \cdot F \to G \cdot F'
\]

are both in \(\mathcal{E}\)

then we get a canonical \(\mathcal{M}\)-grading of \(T\), using

\[
\begin{array}{ccc}
\text{Id} & \xrightarrow{\eta} & T \\
\downarrow{J} & & \downarrow{\mu} \\
T & \xrightarrow{} & T
\end{array}
\]

\[
\begin{array}{c}
J X = \{\eta x \mid x \in X\} \\
(S_1 \sqcap S_2)X = \{\mu t \mid t \text{ is in the image of } S_1(S_2X) \hookrightarrow T(TX)\}
\end{array}
\]

We can do something similar if we replace \([\text{Set}, \text{Set}]\) with some other monoidal category
Grading by sets of shapes

Every monad $T$ on $\text{Set}$ has a set of shapes $T_1$, and every $t \in TX$ has a shape $T!t \in T_1$

- e.g. $\text{List}_1 \cong \mathbb{N}$, and $T!t$ is the length of $t \in \text{List} X$

Whenever $T$ is cartesian, subsets $\Sigma \subseteq T_1$ form a canonical cartesian grading of $T$:

- subsets $\Sigma \subseteq T_1$ are equivalently cartesian subfunctors $S \subseteq T$, via
  $$\Sigma = S_1 \quad \text{SX} = \{ t \in TX \mid T!t \in \Sigma \}$$

  (S is automatically cartesian)

- up to isomorphism, cartesian subfunctors form a factorization system $(\mathcal{E}, \mathcal{M})$ on $[\text{Set}, \text{Set}]_{\text{cart}}$

  \[ \begin{array}{ccc}
  F1 & \rightarrow & G1 \\
  \downarrow S1 & & \downarrow S1 \\
  \end{array} \]
Grading by sets of shapes

When $T = \text{List}$:

- Cartesian subfunctors $S \hookrightarrow \text{List}$ are equivalently subsets
  
  $\Sigma \subseteq \mathbb{N}$

  (allowable lengths of lists)

- which form an ordered monoid with
  
  $\Sigma \leq \Sigma' \iff \Sigma \subseteq \Sigma'$
  
  $J = \{0\}$
  
  $\Sigma_1 \boxdot \Sigma_2 = \{n_1 + n_2 \mid n_1 \in \Sigma_1, n_2 \in \Sigma_2\}$

- The canonical cartesian grading of List is
  
  $\text{List}'\Sigma X = \{xs \in \text{List}X \mid \text{length} \; xs \in \Sigma\}$
Algebraic operations

An algebraic operation for $T$ is a function \[ T^n \rightarrow T \] compatible with the multiplication of $T$

Given subfunctors $S_1, \ldots, S_n$ of $T$, we get

- a canonical subfunctor $S'$
- a flexibly graded algebraic operation

\[ \hat{T}(S_1 \Box -) \times \cdots \times \hat{T}(S_n \Box -) \rightarrow \hat{T}(S' \Box -) \]

(under some extra conditions about products)
Given a suitable class $\mathcal{M}$ of natural transformations, every monad $T$ has a **canonical $\mathcal{M}$-grading** $\hat{T} : \mathcal{M}/T \to [C, C]$

In particular, each monad $T$ on $\text{Set}$ has canonical gradings by

- subfunctors $S \hookrightarrow T$
- subsets $S_1 \subseteq T_1$ (assuming $T$ is cartesian)