Canonical gradings of monads

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Example

The writer monad Wr for lists over a set C has:

object mapping	$\mathrm{Wr} : \mathbf{Set} \longrightarrow \mathbf{Set}$	$\mathrm{Wr}\, X = \mathrm{List}\, C \times X$
unit functions	$\eta_X : X \longrightarrow \mathrm{Wr}\, X$	$\eta_X x = ([\,], x)$
multiplication functions	$\mu_X : \mathrm{Wr}\,(\mathrm{Wr}\, X) \longrightarrow \mathrm{Wr}\, X$	$\mu_X(s_1, (s_2, x)) = (s_1 \mathbin{+\!\!+} s_2, x)$

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We can grade this by

- natural numbers $e \in \mathbf{N}$, to get a graded monad WrL:

$$\mathrm{WrL}\, e\, X = \mathrm{List}_{\leq e} C \times X \qquad \eta : X \longrightarrow \mathrm{WrL}\, 0\, X \qquad \mu : \mathrm{WrL}\, e_1\, (\mathrm{WrL}\, e_2\, X) \longrightarrow \mathrm{WrL}(e_1 + e_2)\, X$$

- subsets $e \subseteq C$, to get a graded monad WrS:

$$\mathrm{WrS}\, e\, X = \mathrm{List}\, e \times X \qquad \eta : X \longrightarrow \mathrm{WrS}\, \emptyset\, X \qquad \mu : \mathrm{WrS}\, e_1\, (\mathrm{WrS}\, e_2\, X) \longrightarrow \mathrm{WrS}(e_1 \cup e_2)\, X$$

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We can grade this by

- subsets $\Sigma \subseteq \mathrm{List}\, C$, to get a graded monad WrC:

$$\mathrm{WrC}\, \Sigma\, X = \Sigma \times X \qquad \eta : X \longrightarrow \mathrm{WrC}\, \mathsf{J}\, X \qquad \mu : \mathrm{WrC}\, \Sigma_1\, (\mathrm{WrC}\, \Sigma_2\, X) \longrightarrow \mathrm{WrC}(\Sigma_1 \boxdot \Sigma_2)\, X$$

where

$$\mathsf{J} = \{[]\} \qquad \Sigma_1 \boxdot \Sigma_2 = \{s_1 +\!\!+\, s_2 \mid s_1 \in \Sigma_1, s_2 \in \Sigma_2\}$$

Example

WrC is the canonical grading of Wr:

- WrL is

$$\mathbf{N} \xrightarrow{F} \mathcal{P}(\mathrm{List}\, C) \xrightarrow{\mathrm{WrC}} [\mathbf{Set}, \mathbf{Set}]$$

  where

$$Fe = \mathrm{List}_{\leq e} C \quad \subseteq \quad \mathrm{List} C$$

- WrS is

$$\mathcal{P} C \xrightarrow{F} \mathcal{P}(\mathrm{List}\, C) \xrightarrow{\mathrm{WrC}} [\mathbf{Set}, \mathbf{Set}]$$

  where

$$Fe = \mathrm{List} e \quad \subseteq \quad \mathrm{List} C$$

This work

For every monad T on **Set**:

- there is a notion of grading of T
- T has a canonical grading
- every other grading factors through the canonical one

This work

More generally, given a suitable notion of $\mathcal{M}$-subfunctor, for every monad T:

- there is a notion of $\mathcal{M}$-grading of T
- T has a canonical $\mathcal{M}$-grading
- every other $\mathcal{M}$-grading factors through the canonical one

In particular, monads can be canonically graded by shapes

Gradings

Let T be a monad on **Set**

$$T : \mathbf{Set} \to \mathbf{Set} \qquad \eta : X \to TX \qquad \mu : T(TX) \to TX$$

A grading $(\mathcal{E}, G)$ of T consists of:

- an partially ordered monoid $(\mathcal{E}, \leq, I, \odot)$ of grades $e \in \mathcal{E}$
- a subset $GeX \subseteq TX$ for each $e \in \mathcal{E}$ and set X

such that

- $GeX \subseteq Ge'X$ for all $e \leq e'$ and X
- G is closed under the monad structure of T:

$Tf : TX \to TY$	restricts to	$Gef : GeX \to GeY$	for each $e \in \mathcal{E}$ and $f : X \to Y$
$\eta : X \to TX$	restricts to	$\eta : X \to GIX$	for each X
$\mu : T(TX) \to TX$	restricts to	$\mu : Ge_1(Ge_2X) \to G(e_1 \odot e_2)X$	for each $e_1, e_2 \in \mathcal{E}$ and X

Fact: (G, η, μ) is a graded monad

Gradings

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such that some conditions hold.

- $(\mathcal{E}, \leq, I, \odot) = (\mathbf{N}, \leq, 0, +)$, and

$$\mathrm{WrL}\, e\, X \;=\; \mathsf{List}_{\leq e} C \times X \;\subseteq\; \mathsf{List}\, C \times X \;=\; \mathrm{Wr}\, X$$

- $(\mathcal{E}, \leq, I, \odot) = (\mathcal{P}C, \subseteq, \emptyset, \cup)$, and

$$\mathrm{WrS}\, e\, X \;=\; \mathsf{List}\, e \times X \;\subseteq\; \mathsf{List}\, C \times X \;=\; \mathrm{Wr}\, X$$

- $(\mathcal{E}, \leq, I, \odot) = (\mathcal{P}(\mathsf{List} C), \subseteq, \emptyset, \cup)$, and

$$\mathrm{WrS}\, e\, X \;=\; \mathsf{List}\, e \times X \;\subseteq\; \mathsf{List}\, C \times X \;=\; \mathrm{Wr}\, X$$

Aside: constructing the ordered monoid

Given

- a set $\mathcal{E}$
- a subfunctor $Ge \subseteq T$ for each $e \in \mathcal{E}$

under some conditions we get a grading of T by defining

$$\begin{array}{rcl} e \leq e' & \Leftrightarrow & Ge \subseteq Ge' \\ I & = & \text{smallest } e \text{ such that } \eta \text{ restricts to } \eta : X \longrightarrow GeX \\ e_1 \odot e_2 & = & \text{smallest } e \text{ such that } \mu \text{ restricts to } \mu : Ge_1(Ge_2X) \longrightarrow GeX \end{array}$$

Canonical grading of a functor

A grading $(\mathcal{E}, G)$ of a functor T consists of:

- a partially ordered set $(\mathcal{E}, \leq)$ of grades $e \in \mathcal{E}$
- a subfunctor $Ge \subseteq T$ for each $e \in \mathcal{E}$

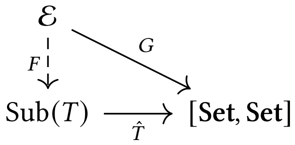
such that $Ge \subseteq Ge'$ for all $e \leq e'$

The canonical grading $(\mathrm{Sub}(T), \hat{T})$ of T has: (ignoring some size issues)

poset	$\mathrm{Sub}(T)$	(subfunctors of T, ordered by pointwise inclusion)
subfunctors	$\hat{T}S = S$	

Universal property:

for every other grading $(\mathcal{E}, G)$ of T, there is a unique monotone $F : \mathcal{E} \to \mathrm{Sub}(T)$ such that $Ge = \hat{T}(Fe)$ for all e



Canonical grading of a monad

The canonical grading $(\mathrm{Sub}(\mathsf{T}), \hat{\mathsf{T}})$ of a monad T has

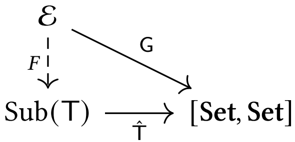
ordered monoid $(\mathrm{Sub}(T), \mathsf{J}, \boxdot)$
subfunctors $\hat{T}S = S$

where

$$\mathsf{J}X = \{\eta x \mid x \in X\}$$
$$(S_1 \boxdot S_2)X = \{\mu\, t \mid t \text{ is in the image of } S_1(S_2X) \hookrightarrow T(TX)\}$$

Universal property:

for every other grading $(\mathcal{E}, \mathsf{G})$ of T, there is a unique $F : \mathcal{E} \to \mathrm{Sub}(T)$ that is lax monoidal and satisfies $Ge = \hat{T}(Fe)$ for all e



$$e \le e' \Rightarrow Fe \subseteq Fe'$$
$$\mathsf{J} \subseteq FI$$
$$Fe \boxdot Fe' \subseteq F(e \odot e')$$

Example: writer

Take the writer monad Wr

$$\mathrm{Wr}\,X = \mathrm{List}C \times X$$

Subfunctors $S \subseteq \mathrm{Wr}$ are equivalently subsets

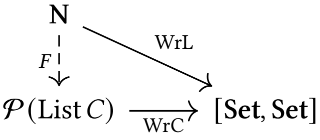
$$\Sigma \subseteq \mathrm{List}C$$

via

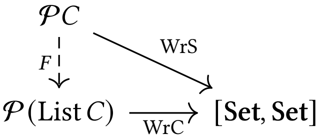
$$\Sigma = \{s \in \mathrm{List}C \mid (s, \star) \in S1\}$$
$$SX = \{(s, x) \in \mathrm{List}C \times X \mid s \in \Sigma\}$$

So the canonical grading is $\mathcal{P}(\mathrm{List}C)$ with

$$\mathrm{WrC}\,\Sigma\,X = \Sigma \times X$$
$$\Sigma \leq \Sigma' \Leftrightarrow \Sigma \subseteq \Sigma' \quad \mathsf{J} = \{[\,]\}$$
$$\Sigma_1 \boxdot \Sigma_2 = \{s_1 +\!\!+ s_2 \mid s_1 \in \Sigma_1, s_2 \in \Sigma_2\}$$



$$Fe = \{s \in \mathrm{List}C \mid |s| \leq e\}$$



$$Fe = \mathrm{List}\,e$$

Example: reader

Take the reader monad Read_V (for a set V)

$$\mathrm{Read}_V\, X \;=\; V \Rightarrow X$$

Subfunctors $S \subseteq \mathrm{Read}_V$ are equivalently <u>upwards-closed</u> sets

$$\Sigma \subseteq \mathrm{Equiv}_V$$

$R \in \Sigma \Rightarrow R' \in \Sigma$ whenever $R \subseteq R'$

of equivalence relations of V, via

$$\Sigma = \{R \in \mathrm{Equiv}_V \mid [-]_R \in S(V/R)\}$$

$$SX = \{f : V \to X \mid \exists R \in \Sigma.\, \forall v, v'.\, v\, R\, v' \Rightarrow f v = f v'\}$$

and these give a canonical grading $(\mathrm{Sub}(\mathrm{Read}_V), \mathsf{ReadC}_V)$

Example: for $FR = \{R' \in \mathrm{Equiv}_V \mid R \subseteq R'\}$, the graded monad

$$\mathrm{Equiv}_V \xrightarrow{F} \mathrm{Sub}(\mathrm{Read}_V) \xrightarrow{\mathsf{ReadC}_V} [\mathbf{Set}, \mathbf{Set}]$$

is

$$\mathrm{Read}'_V\, R\, X \;\cong\; (V/R) \Rightarrow X$$

$\mathcal{M}$-gradings of functors

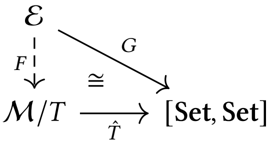
For a class $\mathcal{M}$ of natural transformations $\rightarrowtail$, an $\mathcal{M}$-grading of a functor T consists of

- a category $\mathcal{E}$
- a functor $G : \mathcal{E} \to [\mathbf{Set}, \mathbf{Set}]$
- a natural transformation $m_e : Ge \rightarrowtail T$, whose components are in $\mathcal{M}$

The $\mathcal{M}$-subfunctors of T form an $\mathcal{M}$-grading $(\mathcal{M}/T, \hat{T})$, with

$$\hat{T}(S, m) = S \overset{m}{\rightarrowtail} T$$

and this is canonical:



(more precisely, it is pseudoterminal in the 2-category of $\mathcal{M}$-gradings of T)

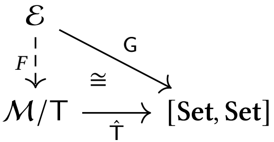
Example: $\mathcal{M}$ = componentwise injective

$\mathcal{M}$-gradings of monads

For a class $\mathcal{M}$ of natural transformations $\rightarrowtail$, an $\mathcal{M}$-grading of a monad T consists of

- a monoidal category $\mathcal{E}$
- a graded monad $G : \mathcal{E} \to [\mathbf{Set}, \mathbf{Set}]$
- a monoidal natural transformation $m_e : Ge \rightarrowtail T$, whose components are in $\mathcal{M}$

Under some conditions, the canonical $\mathcal{M}$-grading of the functor T gives a canonical $\mathcal{M}$-grading of the monad T



(more precisely, it is pseudoterminal in the 2-category of $\mathcal{M}$-gradings of T)

$\mathcal{M}$-gradings of monads

If

- $\mathcal{M}$ forms a factorization system $(\mathcal{E}, \mathcal{M})$ on $[\mathbf{Set}, \mathbf{Set}]$

$$F \xrightarrow{f} G \;=\; F \overset{e}{\twoheadrightarrow} S \overset{m}{\rightarrowtail} G \qquad (e \in \mathcal{E}, m \in \mathcal{M})$$

- for each $e : F \twoheadrightarrow F'$ in $\mathcal{E}$ and G,

$$(e \cdot G) : F \cdot G \longrightarrow F' \cdot G \qquad (G \cdot e) : G \cdot F \longrightarrow G \cdot F'$$

  are both in $\mathcal{E}$

then we get a canonical $\mathcal{M}$-grading of T, using

$$\mathrm{Id} \xrightarrow{\eta} T \;=\; \mathrm{Id} \twoheadrightarrow \mathsf{J} \rightarrowtail T \qquad S_1 \cdot S_2 \longrightarrow T \cdot T \xrightarrow{\mu} T \;=\; S_1 \cdot S_2 \twoheadrightarrow S_1 \boxdot S_2 \rightarrowtail T$$

$$\mathsf{J}X = \{\eta x \mid x \in X\} \qquad (S_1 \boxdot S_2)X = \{\mu\, t \mid t \text{ is in the image of } S_1(S_2X) \hookrightarrow T(TX)\}$$

We can do something similar if we replace $[\mathbf{Set}, \mathbf{Set}]$ with some other monoidal category

Grading by sets of shapes

Every monad T on **Set** has a set of shapes $T1$, and every $t \in TX$ has a shape $T!t \in T1$

- e.g. $\mathrm{List}1 \cong \mathbf{N}$, and $T!t$ is the length of $t \in \mathrm{List}\,X$

Whenever T is cartesian, subsets $\Sigma \subseteq T1$ form a canonical cartesian grading of T:

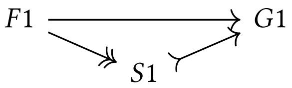
- subsets $\Sigma \subseteq T1$ are equivalently cartesian subfunctors $S \subseteq T$, via

$$\Sigma = S1 \qquad SX = \{t \in TX \mid T!t \in \Sigma\}$$

families of subsets $SX \subseteq TX$ closed under $Tf : TX \to TY$ satisfying $SX = \{t \mid T!t \in S1\}$

(S is automatically cartesian)

- up to isomorphism, cartesian subfunctors form a factorization system $(\mathcal{E}, \mathcal{M})$ on $[\mathbf{Set}, \mathbf{Set}]_{\mathrm{cart}}$



Grading by sets of shapes

When T = List:

- Cartesian subfunctors $S \hookrightarrow \mathrm{List}$ are equivalently subsets

$$\Sigma \subseteq \mathbf{N}$$

  (allowable lengths of lists)

- which form an ordered monoid with

$$\Sigma \leq \Sigma' \Leftrightarrow \Sigma \subseteq \Sigma' \qquad \mathsf{J} = \{0\} \qquad \Sigma_1 \boxdot \Sigma_2 = \{n_1 + n_2 \mid n_1 \in \Sigma_1, n_2 \in \Sigma_2\}$$

- The canonical cartesian grading of List is

$$\mathrm{List}' \Sigma X = \{\mathrm{xs} \in \mathrm{List} X \mid \mathrm{length}\,\mathrm{xs} \in \Sigma\}$$

Algebraic operations

An algebraic operation for T is a function [Plotkin and Power '03]

$$T^n \longrightarrow T$$

compatible with the multiplication of T

Given subfunctors $S_1, \ldots, S_n$ of T, we get

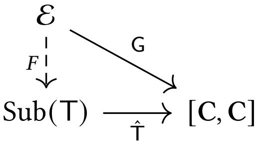
- a canonical subfunctor S'
- a flexibly graded algebraic operation

$$\hat{T}(S_1 \boxdot -) \times \cdots \times \hat{T}(S_n \boxdot -) \longrightarrow \hat{T}(S' \boxdot -)$$

(under some extra conditions about products)

Summary

Given a suitable class $\mathcal{M}$ of natural transformations, every monad T has a canonical $\mathcal{M}$-grading $\hat{T} : \mathcal{M}/\mathsf{T} \to [\mathbb{C}, \mathbb{C}]$



In particular, each monad T on **Set** has canonical gradings by

- subfunctors $S \hookrightarrow T$
- subsets $S1 \subseteq T1$ (assuming T is cartesian)