Grading call-by-push-value, explicitly and implicitly

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Two developments in computational effects

- ***Grading*** (Katsumata 2014, and others)
  Static analysis of computational effects
- ***Call-by-push-value*** (Levy 1999)
  A calculus for studying computational effects

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Grading

A paradigm for static analysis of effectful programs

1. Choose a collection of *grades* e
2. Instantiate language-specific inference rules, to associate a grade to each effectful syntactic element
3. Prove properties of "computations of grade e"

Grading example: type-and-effect analysis

"has grade $e \subseteq \{\text{get}, \text{put}, \text{raise}, \ldots\}$"	means	"does not use any operation that is not in e"

```
x ← get() ;
if x then raise() else return x
```

has grade $\{\text{get}, \text{raise}\}$

```
x ← get() ;
put(true) ;
if x then raise() else return x
```

does not have grade $\{\text{get}, \text{raise}\}$

Grading example: type-and-effect analysis

"has grade $e \subseteq \{\text{get}, \text{put}, \text{raise}, \ldots\}$"	means	"does not use any operation that is not in e"

$$x \leftarrow \texttt{get}() \,;\ \texttt{if}\ x\ \texttt{then}\ \texttt{raise}()\ \texttt{else}\ \texttt{return}\ x$$

has grade $\{\text{get}, \text{raise}\}$

Some of the inference rules:

$$\frac{}{\texttt{return}(v)\ \text{has grade}\ \{\}} \qquad \frac{t\ \text{has grade}\ d \qquad u\ \text{has grade}\ e}{(x \leftarrow t\,;\ u)\ \text{has grade}\ (d \cup e)} \qquad \frac{t\ \text{has grade}\ d \qquad d \subseteq e}{t\ \text{has grade}\ e}$$

Grading

Assume that:

- e are elements of an ordered monoid

$$(\mathbb{E}, \leqslant, \mathbf{1}, \cdot)$$

$$\frac{}{\texttt{return}(v) \text{ has grade } \mathbf{1}} \qquad \frac{t \text{ has grade } d \qquad u \text{ has grade } e}{(\, x \leftarrow t \,;\; u\,) \text{ has grade } (d \cdot e)} \qquad \frac{t \text{ has grade } d \qquad d \leqslant e}{t \text{ has grade } e}$$

- Each effect-causing operation has an associated grade

Call-by-push-value (without grades)

Split syntax into *values* $V, W : A, B$ and *computations* $M, N : \underline{C}, \underline{D}$

$\underline{C}, \underline{D} ::= \mathbf{F}A$		*returners*: running $M : \mathbf{F}A$ may have effects, and any result has type A
	$\mid A \to \underline{C}$	*functions*: application of $M : A \to \underline{C}$ to $V : A$ has type $\underline{C}$
	$\mid \prod_{i \in I} \underline{C}_i$	*tuples*: the ith projection of $M : \prod_{i \in I} \underline{C}_i$ has type $\underline{C}_i$

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Computations include:

$$\frac{\Gamma \vdash V : A}{\Gamma \vdash \mathbf{return}\, V : \mathbf{F}A} \qquad \frac{\Gamma \vdash M : \mathbf{F}A \qquad \Gamma, x : A \vdash N : \underline{C}}{\Gamma \vdash M \,\mathbf{to}\, x.\, N : \underline{C}}$$

$$\frac{\mathrm{op} : A \rightsquigarrow B \qquad \Gamma \vdash V : A \qquad \Gamma, y : B \vdash M : \underline{C}}{\Gamma \vdash \mathbf{do}\, y \leftarrow \mathrm{op}\, V \,\mathbf{then}\, M : \underline{C}}$$

e.g.
$$\begin{aligned} \mathrm{get} &: \mathbf{1} \rightsquigarrow \mathbf{bool} \\ \mathrm{put} &: \mathbf{bool} \rightsquigarrow \mathbf{1} \\ \mathrm{raise} &: \mathbf{1} \rightsquigarrow \mathbf{empty} \end{aligned}$$

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$$\frac{\Gamma, x : A \vdash M : \underline{C}}{\Gamma \vdash \lambda x : A.\, M : A \to \underline{C}}$$

This work: grading call-by-push-value

Key insights:

1. Grades are for tracking observable effects
2. We observe effects at returner type

$\underline{C}, \underline{D} ::= \mathbf{F}A$		*returners*: ***running*** $M : \mathbf{F}A$ ***may have effects***, and any result has type A
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Call-by-push-value with effects (CBPVE)

$\underline{C}, \underline{D} ::= \mathbf{F}_e\, A$		*returners*: running $M : \mathbf{F}_e A$ may have effects ***of grade e***, and any result has type A
	$\mid A \to \underline{C}$	*functions*: application of $M : A \to \underline{C}$ to $V : A$ has type $\underline{C}$
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Subtyping $A <: B$ and $\underline{C} <: \underline{D}$:

$$\frac{d \leqslant e \qquad A <: B}{\mathbf{F}_d A <: \mathbf{F}_e B}$$

+ congruence rules

Action $\langle\!\langle d \rangle\!\rangle \underline{C}$ of $\mathbb{E}$ on computation types:

$$\langle\!\langle d \rangle\!\rangle (\mathbf{F}_e A) = \mathbf{F}_{d \cdot e} A$$

$$\langle\!\langle d \rangle\!\rangle (A \to \underline{C}) = A \to \langle\!\langle d \rangle\!\rangle \underline{C}$$

$$\langle\!\langle d \rangle\!\rangle \left(\prod_{i \in I} \underline{C}_i\right) = \prod_{i \in I} \langle\!\langle d \rangle\!\rangle \underline{C}_i$$

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Computations include:

$$\frac{\Gamma \vdash^{\mathsf{g}} V : A}{\Gamma \vdash^{\mathsf{g}} \mathbf{return}\, V : \mathbf{F}_{\mathbf{1}} A} \qquad \frac{\Gamma \vdash^{\mathsf{g}} M : \mathbf{F}_d A \qquad \Gamma, x : A \vdash^{\mathsf{g}} N : \underline{C}}{\Gamma \vdash^{\mathsf{g}} M\, \mathbf{to}\, x.\, N : \langle\!\langle d \rangle\!\rangle \underline{C}} \qquad \frac{\Gamma \vdash^{\mathsf{g}} M : \underline{C} \qquad \underline{C} <: \underline{D}}{\Gamma \vdash^{\mathsf{g}} \mathbf{coerce}_{\underline{D}}\, M : \underline{D}}$$

$$\frac{\mathsf{op} : A \rightsquigarrow_d B \qquad \Gamma \vdash^{\mathsf{g}} V : A \qquad \Gamma, y : B \vdash^{\mathsf{g}} M : \underline{C}}{\Gamma \vdash^{\mathsf{g}} \mathbf{do}\, y \leftarrow \mathsf{op}\, V\, \mathbf{then}\, M : \langle\!\langle d \rangle\!\rangle \underline{C}}$$

e.g.

$$\begin{aligned} \mathsf{get} &: \mathbf{1} \rightsquigarrow_{\{\mathsf{get}\}} \mathbf{bool} \\ \mathsf{put} &: \mathbf{bool} \rightsquigarrow_{\{\mathsf{put}\}} \mathbf{1} \\ \mathsf{raise} &: \mathbf{1} \rightsquigarrow_{\{\mathsf{raise}\}} \mathbf{empty} \end{aligned}$$

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Computations include:

$$\frac{\Gamma, x : A \vdash^{g} M : \underline{C}}{\Gamma \vdash^{g} \lambda x : A.\, M : A \to \underline{C}}$$

Example

$$\begin{array}{l} x \leftarrow \mathtt{get}()\,; \\ \mathtt{if}\ x\ \mathtt{then}\ \mathtt{raise}()\ \mathtt{else}\ \mathtt{return}\ x \end{array}$$

has grade {get, raise}

CBPVE computation of type $\mathbf{F}_{\{\text{get},\text{raise}\}}\mathbf{bool}$:

$$\begin{array}{l} \mathbf{do}\ x \leftarrow \text{get}()\ \mathbf{then} \\ \mathbf{match}\ x\ \mathbf{with}\ \{\ \ \text{true}.\ \mathbf{do}\ z \leftarrow \text{raise}()\ \mathbf{then\ match}\ z\ \mathbf{with}\ \{\} \\ \qquad\qquad\qquad ,\ \ \text{false}.\ \mathbf{coerce}_{\mathbf{F}_{\{\text{raise}\}}\mathbf{bool}}\,(\mathbf{return}\ x)\} \end{array}$$

Graded algebra models

We get a denotational semantics from any

- *strong graded monad* T on a bicartesian closed category, equipped with
- a morphism $\kappa_{\mathrm{op}} : [\![A]\!] \to T[\![B]\!]d$ for each $\mathrm{op} : A \rightsquigarrow_d B$

$$
\begin{array}{rcl}
\text{value type } A & \mapsto & \text{object } [\![A]\!] \\
\text{computation type } \underline{C} & \mapsto & T\text{-algebra } [\![\underline{C}]\!] \\
\text{typing context } \Gamma & \mapsto & \text{object } [\![\Gamma]\!] \\
\text{value subtyping } A <: B & \mapsto & \text{morphism } [\![A <: B]\!] : [\![A]\!] \to [\![B]\!] \\
\text{computation subtyping } \underline{C} <: \underline{D} & \mapsto & \text{algebra morphism } [\![\underline{C} <: \underline{D}]\!] : [\![\underline{C}]\!] \to [\![\underline{D}]\!] \\
\text{value } \Gamma \vdash^{\mathsf{g}} V : A & \mapsto & \text{morphism } [\![V]\!] : [\![\Gamma]\!] \to [\![A]\!] \\
\text{computation } \Gamma \vdash^{\mathsf{g}} M : \underline{C} & \mapsto & \text{morphism } [\![M]\!] : [\![\Gamma]\!] \to [\![\underline{C}]\!]_{\mathbf{1}}
\end{array}
$$

Call-by-push-value with effects

A calculus for studying ***graded*** computational effects

- Subsumes graded versions of (fine-grain) call-by-value, and of call-by-name
- Grades are ***explicit*** in the syntax

Grading as analysis

We have a judgment

$$\Gamma \vdash^{\mathtt{g}} M : \underline{C}$$

CBPVE typing context

CBPVE computation

CBPVE computation type

$$\frac{\Gamma \vdash^{\mathtt{g}} M : \underline{C} \qquad \underline{C} <: \underline{D}}{\Gamma \vdash^{\mathtt{g}} \mathbf{coerce}_{\underline{D}}\, M : \underline{D}}$$

$$\frac{\Gamma, x : A \vdash^{\mathtt{g}} M : \underline{C}}{\Gamma \vdash^{\mathtt{g}} \lambda x : A.\, M : A \to \underline{C}}$$

But we want

$$\Gamma \vdash^{\mathtt{i}} M : \underline{C}$$

CBPVE typing context

CBPV computation

CBPVE computation type

$$\frac{\Gamma \vdash^{\mathtt{i}} M : \underline{C} \qquad \underline{C} <: \underline{D}}{\Gamma \vdash^{\mathtt{i}} M : \underline{D}}$$

$$\frac{\Gamma, x : A' \vdash^{\mathtt{i}} M : \underline{C}}{\Gamma \vdash^{\mathtt{i}} \lambda x : A.\, M : A' \to \underline{C}}$$

(where A' is A annotated with grades)

Implicit grades

Define

$$\Gamma \vdash^{\mathtt{i}} M : \underline{C} \quad \text{if} \quad \exists M'.\ \lfloor M' \rfloor = M \wedge \Gamma \vdash^{\mathtt{g}} M' : \underline{C}$$

where

$$\lfloor - \rfloor : \text{CBPVE} \to \text{CBPV}$$

$$\lfloor \mathbf{coerce}_{\underline{D}}\, M \rfloor = \lfloor M \rfloor$$

$$\lfloor \lambda x : A.\, M \rfloor = \lambda x : \lfloor A \rfloor.\ \lfloor M \rfloor$$

erases grades and **coerce**.

Models for implicit grades

If $\Gamma \vdash^{\mathtt{i}} M : \underline{C}$, then define

$$[\![M]\!] = [\![M']\!] \quad \text{where} \quad \lfloor M' \rfloor = M \;\wedge\; \Gamma \vdash^{\mathtt{g}} M' : \underline{C}$$

assuming *coherence*:

$$\lfloor M_1' \rfloor = \lfloor M_2' \rfloor \;\Rightarrow\; [\![M_1']\!] = [\![M_2']\!] \quad \text{for all } \Gamma \vdash^{\mathtt{g}} M_i' : \underline{C}$$

Models for implicit grades

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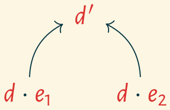
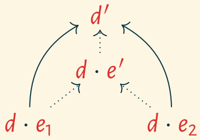
But coherence is *false* in general for graded algebra models.

Proving coherence

Assume the ordered monoid of grades has *left-cancellative upper bounds*:

$$d \cdot e_1 \leqslant d' \geqslant d \cdot e_2 \;\Rightarrow\; \exists e'.\, e_1 \leqslant e' \geqslant e_2 \wedge d \cdot e' \leqslant d'$$





Examples:

- Any join-semilattice such that multiplication left-distributes over joins ($d \cdot (e_1 \sqcup e_2) = (d \cdot e_1) \sqcup (d \cdot e_2)$): take $e' = e_1 \sqcup e_2$
- In particular, $(\mathcal{P}\{\text{get}, \text{put}, \text{raise}, \ldots\}, \subseteq, \{\}, \cup)$

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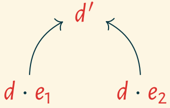

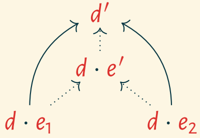

Then coherence holds:

$$\lfloor M_1' \rfloor = \lfloor M_2' \rfloor \;\Rightarrow\; [\![M_1']\!] = [\![M_2']\!] \quad \text{for all } \Gamma \vdash^{g} M_i' : \underline{C}$$

Proof idea.
Use logical relations: relate $\Gamma \vdash^{g} N_1 : \underline{D}_1$ to $\Gamma \vdash^{g} N_2 : \underline{D}_2$, where $\lfloor \underline{D}_1 \rfloor = \lfloor \underline{D}_2 \rfloor$, by $\top\top$-lifting ∎

Three developments in computational effects

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  A calculus for studying computational effects
- ***Call-by-push-value with effects*** (this paper)
  A calculus for studying graded computational effects
  (With implicit grades, assuming coherence)