Galois connecting call-by-value and call-by-name

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Goal

- Call-by-value: \((\lambda x. e) e' \rightsquigarrow^*_v (\lambda x. e) v \rightsquigarrow_v e[x \mapsto v] \rightsquigarrow^*_v \cdots\)

- Call-by-name: \((\lambda x. e) e' \rightsquigarrow^*_n e[x \mapsto e'] \rightsquigarrow^*_n \cdots\)

If we replace call-by-value with call-by-name, then:

- No side-effects: nothing changes
- Only recursion: behaviour changes, but if CBV terminates with result \(v\), CBN terminates with \(v\)
- Only nondeterminism: behaviour also different, but if CBV can terminate with result \(v\), then CBN can also terminate with result \(v\)
- Mutable state: behaviour changes, we can’t say much about how

Questions:

- How can we prove these?
- What properties of the side-effects do we need to prove something?
Goal

- Call-by-value: $(\lambda x. e) e' \rightsquigarrow^*_{v} (\lambda x. e) v \rightsquigarrow_{v} e[x \mapsto v] \rightsquigarrow_{v}^* \ldots$
- Call-by-name: $(\lambda x. e) e' \rightsquigarrow_{n} e[x \mapsto e'] \rightsquigarrow_{n}^* \ldots$

If we replace call-by-value with call-by-name, then:

- No side-effects: nothing changes
- Only recursion: behaviour changes

**CBV:** $(\lambda x. \text{false}) \Omega \rightsquigarrow_{v} (\lambda x. \text{false}) \Omega \rightsquigarrow_{v} \ldots$

**CBN:** $(\lambda x. \text{false}) \Omega \rightsquigarrow_{n} \text{false}$

but if CBV terminates with result $v$, CBN terminates with $v$
Goal

- Call-by-value: $\left(\lambda x . e\right) e' \rightsquigarrow_v^* \left(\lambda x . e\right) v \rightsquigarrow_v e[x \mapsto v] \rightsquigarrow_v^* \cdots$

- Call-by-name: $\left(\lambda x . e\right) e' \rightsquigarrow_n e[x \mapsto e'] \rightsquigarrow_n^* \cdots$

If we replace call-by-value with call-by-name, then:

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How to relate different semantics of the same language

1. Define another language that captures both semantics via two sound and adequate translations \((\cdot)^{\text{v}}, (\cdot)^{\text{n}}\)

\[
\begin{align*}
\text{(CBV)} & \quad (\langle e \rangle)^{\text{v}} \leftrightarrow e \quad \longrightarrow \quad (\langle e \rangle)^{\text{n}} \quad \text{(CBN)}
\end{align*}
\]

5. For programs (closed, ground expressions) \(e\)

\[
(\langle e \rangle)^{\text{v}} \preceq (\langle e \rangle)^{\text{n}}
\]
How to relate different semantics of the same language

1. Define another language that captures both semantics via two sound and adequate translations \( \langle-\rangle^v, \langle-\rangle^n \)

\[
\begin{align*}
\text{(CBV)} & \quad \langle e \rangle^v &\quad \longleftrightarrow & \quad e &\quad \longleftrightarrow & \quad \langle e \rangle^n &\quad \text{(CBN)}
\end{align*}
\]

2. Define maps between the two translations

\[
\begin{align*}
\text{CBV translation of } \tau &\quad \xrightarrow{\Phi_\tau} & \quad \text{CBN translation of } \tau
\end{align*}
\]

3. Show that \( \Phi, \Psi \) satisfy nice properties

4. Relate the two translations of (possibly open) expressions \( e \)

\[
\langle e \rangle^v \leq_{\text{ctx}} \Psi_\tau(\langle e \rangle^n[\Phi_\Gamma])
\]

5. For programs (closed, ground expressions) \( e \)

\[
\langle e \rangle^v \leq \langle e \rangle^n
\]
How to relate different semantics of the same language

To relate CBV and CBN:

1. **Call-by-push-value** [Levy '99] captures CBV and CBN
2. We can define maps $\Phi_\tau, \Psi_\tau$ using the syntax of CBPV
3. When side-effects are (lax) thunkable, these form Galois connections

   $\Phi_\tau \dashv \Psi_\tau$

   (wrt $\leq_{ctx}$)

4. (3) implies $\langle e \rangle^v \leq_{ctx} \Psi_\tau (\langle e \rangle^n [\Phi])$
5. (4) is $\langle e \rangle^v \leq \langle e \rangle^n$ when $e$ is a program
For recursion and nondeterminism, define

\[ M_1 \preceq M_2 \iff \forall V. M_1 \Downarrow \text{return } V \Rightarrow M_2 \Downarrow \text{return } V \quad (\Downarrow \text{ is evaluation in CBPV}) \]

so \( M_1 \preceq_{\text{ctx}} M_2 \) means

\[ \forall V. C[M_1] \Downarrow \text{return } V \Rightarrow C[M_2] \Downarrow \text{return } V \]

for closed, ground contexts \( C \)

Both side-effects are thunkable, so \( \Phi \) and \( \Psi \) form Galois connections, so

\[ \langle e \rangle^v \preceq_{\text{ctx}} \Psi_\tau(\langle e \rangle^n[\Phi_\Gamma]) \]
Example

For programs $e$, we have

$$(e)^v \preceq (e)^n$$

so

$$e \xrightarrow{v}^* v \iff (e)^v \Downarrow \text{return } (v)$$

$$\Rightarrow (e)^n \Downarrow \text{return } (v)$$

$$\iff e \xrightarrow{n}^* v$$

(soundness)

$$(e)^v \preceq (e)^n$$

(adequacy)
Call-by-push-value [Levy ’99]

Split syntax into values and computations

- Values don’t reduce, computations do
Call-by-push-value [Levy '99]

Split syntax into values and computations

- Values don’t reduce, computations do

Evaluation order is explicit

- Sequencing of computations:

\[
\begin{align*}
\Gamma \vdash V : A & \quad \Gamma \vdash M_1 : FA & \quad \Gamma, x : A \vdash M_2 : C \\
\Gamma \vdash \text{return} V : FA & \quad \Gamma \vdash M_1 \text{ to } x. M_2 : C
\end{align*}
\]

- Thunks:

\[
\begin{align*}
\Gamma \vdash M : C & \quad \Gamma \vdash V : UC \\
\Gamma \vdash \text{thunk} M : UC & \quad \Gamma \vdash \text{force} V : C
\end{align*}
\]
**Call-by-value and call-by-name**

Source language types:

\[ \tau ::= \text{unit} \mid \text{bool} \mid \tau \rightarrow \tau' \]

CBV and CBN translations into CBPV:

\[ \begin{align*}
\tau & \mapsto \text{value type } (|\tau|)^v \\
\text{unit} & \mapsto \text{unit} \\
\text{bool} & \mapsto \text{bool} \\
(\tau \rightarrow \tau') & \mapsto U((|\tau|)^v \rightarrow F(|\tau'|)^v) \\
\end{align*} \]

\[ \begin{align*}
\tau & \mapsto \text{computation type } (|\tau|)^n \\
\text{unit} & \mapsto F \text{ unit} \\
\text{bool} & \mapsto F \text{ bool} \\
(\tau \rightarrow \tau') & \mapsto ((U(|\tau|)^n) \rightarrow (|\tau'|)^n) \\
\end{align*} \]

\[ \begin{align*}
\Gamma, x : \tau & \mapsto (|\Gamma|^v, x : (|\tau|^v) \\
\Gamma, x : \tau & \mapsto (|\Gamma|^n, x : U(|\tau|^n) \\
\Gamma \vdash e : \tau & \mapsto (|\Gamma|^v) \vdash (|e|^v : F(|\tau|^v) \\
\Gamma \vdash e : \tau & \mapsto (|\Gamma|^n) \vdash (|e|^n : (|\tau|^n)
\end{align*} \]
Call-by-value and call-by-name

Define maps between CBV and CBN:

$$\Gamma \vdash M : F \langle \tau \rangle^v \quad \Rightarrow \quad \Gamma \vdash \Phi \tau M : \langle \tau \rangle^n$$  \hspace{1cm} (CBV to CBN)

$$\Gamma \vdash N : \langle \tau \rangle^n \quad \Rightarrow \quad \Gamma \vdash \Psi \tau N : F \langle \tau \rangle^v$$  \hspace{1cm} (CBN to CBV)
Call-by-value and call-by-name

Define maps between CBV and CBN:

\[
\Gamma \vdash M : F (\tau)^V \quad \leftrightarrow \quad \Gamma \vdash \Phi_{\tau} M : (\tau)^n \quad \text{(CBV to CBN)}
\]

\[
\Gamma \vdash N : (\tau)^n \quad \leftrightarrow \quad \Gamma \vdash \Psi_{\tau} N : F (\tau)^V \quad \text{(CBN to CBV)}
\]

Example: for \( \tau = \text{unit} \to \text{unit} \), we have

\[
(\text{unit} \to \text{unit})^V = U (\text{unit} \to F \text{unit})
\]

\[
(\text{unit} \to \text{unit})^n = U (F \text{unit}) \to F \text{unit}
\]

\[
M \xrightarrow{\Phi_{\text{unit} \to \text{unit}}} M \text{ to } f. \lambda x. \text{force } x \text{ to } z. z \text{' force } f
\]

\[
N \xrightarrow{\Psi_{\text{unit} \to \text{unit}}} \text{return (thunk } (\lambda x. (\text{thunk return } x) \text{' } N))
\]
Lemma

If \((\Phi_\tau, \Psi_\tau)\) is a Galois connection (adjunction) for each \(\tau\), i.e.

\[
M \preceq_{\text{ctx}} \Psi_\tau(\Phi_\tau M) \quad \Phi_\tau(\Psi_\tau N) \preceq_{\text{ctx}} N
\]

then

\[
\lceil e \rceil^v \preceq_{\text{ctx}} \Psi_\tau(\lceil e \rceil^n[\Phi_\Gamma])
\]
Galois connection between CBV and CBN?

These do not always hold!

\[ M \preceq_{\text{ctx}} \Psi_{\tau}(\Phi_{\tau}M) \quad \Phi_{\tau}(\Psi_{\tau}N) \preceq_{\text{ctx}} N \]

- Don’t hold for: exceptions, mutable state

\[ \text{raise} \not\preceq_{\text{ctx}} \text{return}(\ldots) = \Psi_{\text{unit}\rightarrow\text{unit}}(\Phi_{\text{unit}\rightarrow\text{unit}} \text{raise}) \]

- Do hold for: no side-effects, recursion, nondeterminism

This is where the side-effects matter
Galois connection between CBV and CBN?

Definition (Thunkable [Führmann ’99])

A computation \( \Gamma \vdash M : FA \) is (lax) *Thunkable* if

\[
M \quad \text{to} \quad x. \text{return} (\text{thunk} (\text{return} x)) \quad \leq_{\text{ctx}} \quad \text{return} (\text{thunk} M)
\]

- Essentially: we’re allowed to suspend the computation \( M \)
- Implies \( M \) commutes with other computations, is (lax) discardable, (lax) copyable
Definition (Thunkable [Führmann ’99])

A computation $\Gamma \vdash M : FA$ is (lax) thunkable if

$$M \text{ to } x. \text{return } (\text{thunk } (\text{return } x)) \preceq_{\text{ctx}} \text{return } (\text{thunk } M)$$

- Essentially: we’re allowed to suspend the computation $M$
- Implies $M$ commutes with other computations, is (lax) discardable, (lax) copyable

Lemma

*If every computation is thunkable, then $(\Phi_\tau, \Psi_\tau)$ is a Galois connection.*

---

Galois connection between CBV and CBN?
If every computation is thunkable then

\[(\langle e \rangle)^v \leq_{\text{ctx}} \Psi_\tau(\langle e \rangle^n[\Gamma])\]

for each \(e\). (And the converse holds for computations of ground type.)

And if \(e\) is a program then

\[(\langle e \rangle)^v \leq (\langle e \rangle)^n\]
Denotational semantics

Given an order-enriched model of CBPV

- cartesian closed Poset-category
- coproduct $1 + 1$
- strong Poset-monad $T$

prove that if

- $T$ is lax idempotent ($T\eta_X \sqsubseteq \eta_{TX}$)

then

$$\left[ \langle e \rangle^\nu \right] \sqsubseteq \psi_\tau \circ \left[ \langle e \rangle^n \right] \circ \phi_\Gamma$$

For example:

<table>
<thead>
<tr>
<th>[[\Gamma]]</th>
<th>T</th>
<th>[[M]]</th>
<th>[[M]] \sqsubseteq [[N]]</th>
</tr>
</thead>
<tbody>
<tr>
<td>No side-effects set Id function equality</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Recursion (\omega)cpo (-)(\perp) continuous function pointwise</td>
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</tr>
<tr>
<td>Nondeterminism poset free join-semilattice monotone function pointwise</td>
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- cartesian closed Poset-category
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prove that if

- \( T \) is lax idempotent \((T\eta_X \sqsubseteq \eta_TX)\)

then

\[
\llbracket (e)^v \rrbracket \sqsubseteq \psi_\tau \circ \llbracket (e)^n \rrbracket \circ \phi_\Gamma
\]

If the model is adequate:

\[
\llbracket M_1 \rrbracket \sqsubseteq \llbracket M_2 \rrbracket \Rightarrow M_1 \preceq_{ctx} M_2
\]

then

\[
(e)^v \preceq_{ctx} \Psi_\tau((e)^n[\Phi_\Gamma])
\]
How to relate two different semantics:

1. Translate from source language to intermediate language
2. Define maps between two translations
3. Relate terms:
   \[ \langle e \rangle^V \leq_{\text{ctx}} \Psi_\tau(\langle e \rangle^n[\Phi_\Gamma]) \]

- Works for call-by-value and call-by-name
- Also works for other things like comparing direct and continuation-style semantics [Reynolds '74], strict and lazy products, etc.