Flexible presentations of graded monads

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How can we model quantitative computational effects, where each computation M has a grade $d \in \mathbb{N}$?

$$\frac{\Gamma \vdash V : A}{\Gamma \vdash \mathrm{return}\, V : A \,\&\, 1}$$

$$\frac{}{\Gamma \vdash \mathrm{fail} : A \,\&\, 0} \qquad \frac{\Gamma \vdash M_1 : A \,\&\, d_1 \qquad \Gamma \vdash M_2 : A \,\&\, d_2}{\Gamma \vdash \mathrm{or}(M_1, M_2) : A \,\&\, (d_1 + d_2)}$$

- e.g. to prove equations between terms

$$\mathrm{or}(\mathrm{or}(M_1, M_2), M_3) \equiv \mathrm{or}(M_1, \mathrm{or}(M_2, M_3))$$
$$\mathrm{or}(M_1, M_2) \equiv M_2 \qquad (\text{if } \Gamma \vdash M_1 : A \,\&\, 0)$$

Models of effects from presentations

1. Monads model computational effects [Moggi '89]
2. They often come from presentations [Plotkin and Power '02]
3. which induce algebraic operations [Plotkin and Power '03]
4. and provide semantics for effect handlers [Plotkin and Pretnar '09]

Example:

1. Nondeterminism can be modelled using the free monoid monad List
2. which comes from the presentation of monoids

$$\mathsf{m} : 2 \qquad \mathsf{u} : 0$$

$$\mathsf{m}(\mathsf{m}(x, y), z) \equiv \mathsf{m}(x, \mathsf{m}(y, z)) \qquad \mathsf{m}(\mathsf{u}, x) \equiv x \equiv \mathsf{m}(x, \mathsf{u})$$

3. which also induces algebraic operations

$$\mathrm{fail}_X = [\![\mathsf{u}]\!] : \mathbf{1} \to \mathrm{List}\, X \qquad \mathrm{or}_X = [\![\mathsf{m}]\!] : \mathrm{List}\, X \times \mathrm{List}\, X \to \mathrm{List}\, X$$

Presentations of monads

Presentation (Σ, E):

operations $\mathrm{op} : n$ from Σ

+ equations $t \equiv u$ from E

Presentation of monoids:

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Algebra:

carrier set A with
functions $[\![\mathrm{op}]\!] : A^n \longrightarrow A$
satisfying equations

Monoid:

carrier set A with
functions $[\![\mathsf{u}]\!] : 1 \longrightarrow A$, $[\![\mathsf{m}]\!] : A \times A \longrightarrow A$
satisfying unit and associativity eqns

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Free algebra on X:

algebra $(TX, [\![-]\!])$ with
function $\eta_X : X \longrightarrow TX$
satisfying universal property

Free monoid on X:

monoid $(\mathrm{List}\, X, \mathrm{fail}, \mathrm{or})$ with
singleton function $X \longrightarrow \mathrm{List}\, X$
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satisfying universal property

Free algebra monad T:
has the same algebras as the presentation

Free monoid monad List:
has monoids as algebras

Models of quantitative effects from graded presentations

1. Nondeterminism can be modelled using a graded monad List

$$\text{List}\, X = \text{the free graded monoid on the set } X$$

2. which comes from a graded presentation of monoids?
3. which induces graded algebraic operations?

$$\text{or}_{d_1,d_2,X} : \text{List}\, X\, d_1 \times \text{List}\, X\, d_2 \to \text{List}\, X\, (d_1 + d_2) \qquad (d_1, d_2 \in \mathbb{N})$$
$$\text{fail}_X : \mathbf{1} \to \text{List}\, X\, 0$$

$$\frac{}{\Gamma \vdash \text{fail} : A \,\&\, 0} \qquad \frac{\Gamma \vdash M_1 : A \,\&\, d_1 \qquad \Gamma \vdash M_2 : A \,\&\, d_2}{\Gamma \vdash \text{or}(M_1, M_2) : A \,\&\, (d_1 + d_2)}$$

The existing notions of graded presentation [Smirnov '08, Milius et al. '15, Dorsch et al. '19, Kura '20] are not general enough to do this

This work

Develop a notion of flexibly graded presentation

- Every flexibly graded presentation (Σ, E) induces
  - a canonical graded monad $\mathsf{T}_{(\Sigma,E)}$
  - along with a flexibly graded algebraic operation for each operation of the presentation
- Examples like List have computationally natural flexibly graded presentations
- The constructions are mathematically justified by locally graded categories, and a notion of flexibly graded abstract clone

Algebras

Algebraic structures form concrete categories $(\mathbb{C}, U : \mathbb{C} \to \mathbf{Set})$, consisting of:

1. a collection of objects $|\mathbb{C}|$;
2. for each object $\mathsf{A} \in \mathbb{C}$, a carrier set $U\mathsf{A}$
3. for each $\mathsf{A}, \mathsf{B} \in \mathsf{A}$, a set $\mathbb{C}(\mathsf{A}, \mathsf{B})$ of functions $f : U\mathsf{A} \to U\mathsf{B}$, called homomorphisms $f : \mathsf{A} \to \mathsf{B}$;
4. such that homomorphisms are closed under identities and composition

$$\mathrm{id}_{UA} : \mathsf{A} \to \mathsf{A}$$
$$(g \circ f) : \mathsf{A}_1 \to \mathsf{A}_3 \quad \text{for } f : \mathsf{A}_1 \to \mathsf{A}_2, g : \mathsf{A}_2 \to \mathsf{A}_3$$

Examples:

- monoids $\mathsf{A} = (A, \mathsf{m}, \mathsf{u})$, with carrier $U\mathsf{A} = A$, and monoid homomorphisms $f : \mathsf{A} \to \mathsf{B}$;
- also rings, groups, mnemoids, semilattices, ...

Algebras

An isomorphism

$$I : (\mathbb{C}, U) \cong (\mathbb{C}', U')$$

of concrete categories consists of

1. a bijection $I\colon |\mathbb{C}| \cong |\mathbb{C}'|$;
2. such that $U(I\mathsf{A}) = U\mathsf{A}'$ for all A, and

$$f \in \mathbb{C}(\mathsf{A}, \mathsf{A}') \quad \Leftrightarrow \quad f \in \mathbb{C}'(I\mathsf{A}, I\mathsf{A}')$$

   for all functions $f : U\mathsf{A} \to U\mathsf{A}'$.

Presentation algebras

Fix a presentation (Σ, E) consisting of a set $\Sigma(n)$ of n-ary operations for each $n \in \mathbb{N}$, together with a collection of equations

- A (Σ, E)-*algebra* $\mathsf{A} = (A, [\![-]\!])$ is a carrier set $U_{(\Sigma,E)}\mathsf{A} = A$, together with interpretation functions

$$[\![\mathrm{op}]\!] : A^n \to A \quad \text{for each op} \in \Sigma(n)$$

  satisfying the equations
- (Σ, E) is a *presentation* of $(\mathbb{C}, U)$ if

$$(\mathbb{C}, U) \cong (\mathbf{Alg}(\Sigma, E), U_{(\Sigma,E)})$$

Examples:

- $(\mathbf{Mon}, U)$ has a presentation

$$\Sigma(0) = \{\mathsf{u}\} \qquad \Sigma(2) = \{\mathsf{m}\} \qquad \Sigma(n) = \emptyset \quad \text{otherwise}$$

$$\mathsf{m}(\mathsf{u}, x) \equiv x \equiv \mathsf{m}(x, \mathsf{u}) \qquad \mathsf{m}(\mathsf{m}(x, y), z) \equiv \mathsf{m}(x, \mathsf{m}(y, z))$$

- also rings, groups, mnemoids, semilattices, ...

Monad algebras

- A monad T (in Kleisli form, on $\mathbf{Set}$) consists of:
  - a set TX for each set X
  - unit functions $\eta_X : X \longrightarrow TX$
  - Kleisli extension $\dfrac{f : X \longrightarrow TY}{f^\dagger : TX \longrightarrow TY}$

  such that the monad laws hold:

$$f^\dagger \circ \eta_X = f \qquad (\eta_X)^\dagger = \mathrm{id}_{TX} \qquad (g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger$$

- A T-*algebra* $\mathsf{A} = (A, (-)^\ddagger)$ is a carrier set $U_\mathsf{T}\mathsf{A} = A$, together with an extension operation $\dfrac{f : X \longrightarrow A}{f^\ddagger : TX \longrightarrow A}$ such that $f^\dagger \circ \eta_X = f$, $(g^\dagger \circ f)^\ddagger = g^\ddagger \circ f^\dagger$

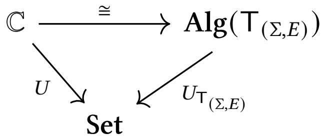
$(\mathbb{C}, U)$ is *monadic* if $(\mathbb{C}, U) \cong (\mathbf{Alg}(\mathsf{T}), U_\mathsf{T})$ for some (unique) monad T

Algebraic structures are monadic

If $(\mathbb{C}, U)$ is any algebraic structure that has a presentation (Σ, E)

- e.g. monoids, rings, groups, arithmoids, semilattices, ...

then $(\mathbb{C}, U)$ is monadic:



Theorem

For a concrete category $(\mathbb{C}, U)$, the following are equivalent:

1. $(\mathbb{C}, U)$ *has a presentation* (Σ, E)*;*
2. $(\mathbb{C}, U)$ *is monadic, and the monad* T *is finitary.*

Grading

Definition

A $(\mathbb{N}_{\leq}-)$*graded set* $X : \mathbb{N}_{\leq} \longrightarrow \mathbf{Set}$ consists of:

- a set Xd for each $d \in \mathbb{N}$
- a function $X(d{\leq}d') : Xd \longrightarrow Xd'$ for each $d \leq d' \in \mathbb{N}$

such that $X(d \leq d) = \mathrm{id}$ and $X(d'{\leq}d'') \circ X(d{\leq}d') = X(d{\leq}d'')$.

A *morphism* $f : X \multimap{e} Y$ of grade $e \in \mathbb{N}$ is a natural family of functions

$$f_d : Xd \longrightarrow Y(d \cdot e)$$

Identities have grade 1, composition multiplies grades, and we can coerce a morphism to a larger grade:

$$\begin{aligned} \mathrm{id}_X &: X -1\!\to X \\ (g \circ f) &: X_1 -e \cdot e'\!\to X_3 \quad \text{for } f : X_1 -e\!\to X_2, g : X_2 -e'\!\to X_3 \\ (e \leq e')^* f &: X -e'\!\to Y \quad \text{for } f : X -e\!\to Y \\ &\qquad \text{where } ((e \leq e')^* f)_d x = Y(d{\cdot}e \leq d{\cdot}e')(f_d x) \end{aligned}$$

so we get a locally graded category [Wood '76] of graded sets

Grading

- For each (ungraded) set X, there is a graded set $\mathrm{List}\,X$:
  - $\mathrm{List}Xd$ is lists over X of length $\leq d$
  - $\mathrm{List}X(d \leq d')$ is the inclusion $\mathrm{List}Xd \subseteq \mathrm{List}Xd'$

  and morphism dup : $\mathrm{List}X \multimap 2 \to \mathrm{List}X$

$$\mathsf{dup}_d : \mathrm{List}Xd \longrightarrow \mathrm{List}X(d \cdot 2)$$
$$\mathsf{dup}_d[x_1, x_2, \ldots, x_k] = [x_1, x_1, x_2, x_2, \ldots, x_k, x_k]$$

- Every (ungraded) set X forms a graded set KX such that morphisms $f : KX - e \to Y$ are equivalently functions $f_1 : X \longrightarrow Ye$:

$$KXd = \begin{cases} X & \text{if } d \geq 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Graded algebraic structures

A *graded monoid* $\mathsf{A} = (A, m, u)$ consists of:

- a graded set A (the carrier)
- multiplication functions $m_{d_1,d_2} : Ad_1 \times Ad_2 \longrightarrow A(d_1 + d_2)$ natural in $d_1, d_2 \in \mathbb{N}_{\leq}$
- a unit $u \in A0$

such that

$$m_{0,d}(u, x) = x = m_{d,0}(x, u)$$

$$m_{d_1+d_2,d_3}(m_{d_1,d_2}(x, y), z) = m_{d_1,d_2+d_3}(x, m_{d_2,d_3}(y, z))$$

A *morphism* $f : \mathsf{A} -e\!\to \mathsf{B}$ of grade e is a graded set morphism $f : A -e\!\to B$ such that

$$f_{d_1+d_2}(m_{d_1,d_2}(x_1, x_2)) = m_{d_1\cdot e,d_2\cdot e}(f_{d_1}x_1, f_{d_2}x_2) \qquad f_0\mathsf{u} = \mathsf{u}$$

Example: the free graded monoid on a set X is

- graded set $\mathrm{List}X$, with
- concatenation of lists $\mathrm{List}Xd_1 \times \mathrm{List}Xd_2 \longrightarrow \mathrm{List}X(d_1 + d_2)$
- the empty list $[] \in \mathrm{List}X0$

Graded algebraic structures

Graded algebraic structures form concrete locally graded categories $(\mathcal{C}, U : \mathcal{C} \to \mathbf{GSet})$, consisting of:

1. a collection of objects $|\mathcal{C}|$;
2. for each object $\mathsf{A} \in \mathcal{C}$, a carrier graded set $U\mathsf{A}$
3. for each $\mathsf{A}, \mathsf{B} \in \mathsf{A}$, and grade e, a set $\mathcal{C}(\mathsf{A}, \mathsf{B})e$ of morphisms $f : U\mathsf{A} -e\!\!\to U\mathsf{B}$, the morphisms $f : \mathsf{A} -e\!\!\to \mathsf{B}$ of grade e;
4. such that morphisms are closed under identities, composition, and coercions

$$\begin{aligned} \mathrm{id}_{UA} &: \mathsf{A} -1\!\!\to \mathsf{A} \\ (g \circ f) &: \mathsf{A}_1 -e \cdot e'\!\!\to \mathsf{A}_3 \quad \text{for } f : \mathsf{A}_1 -e\!\!\to \mathsf{A}_2, g : \mathsf{A}_2 -e'\!\!\to \mathsf{A}_3 \\ (e \leq e')f^* &: \mathsf{A} -e'\!\!\to \mathsf{B} \quad \text{for } e \leq e', f : \mathsf{A} -e\!\!\to \mathsf{B} \end{aligned}$$

Examples:

- graded monoids $\mathsf{A} = (A, \mathrm{m}, \mathrm{u})$, with carrier $U\mathsf{A} = A$;
- also graded rings, graded modules, ...

Graded presentations

[Smirnov '08, Milius et al. '15, Dorsch et al. '19, Kura '20]

Fix a (rigidly) graded presentation (Σ, E) consisting of a set $\Sigma(n, d)$ of n-ary operations of grade d for each $n, d \in \mathbb{N}$, together with a collection of equations

- A (Σ, E)-*algebra* $\mathsf{A} = (A, [\![-]\!])$ is a graded set $U_{(\Sigma,E)}\mathsf{A} = A$, together with interpretation functions

$$[\![\mathrm{op}]\!]_e : (Ae)^n \to A(d \cdot e) \quad \text{for each op} \in \Sigma(n, d)$$

  satisfying the equations
- (Σ, E) is a *presentation* of (C, U) if

$$(C, U) \cong (\mathbf{Alg}(\Sigma, E), U_{(\Sigma,E)})$$

Graded monads

[Borceux, Janelidze, Kelly '05; Smirnov '08; Melliès '12; Katsumata '14]

- A *graded monad* T consists of:
  - a graded set TX for each (ungraded) set X
  - unit functions $\eta_X : X \to TX1$
  - Kleisli extension $\dfrac{f : X \to TYe}{f^\dagger_d : TXd \to TY(d \cdot e)}$ natural in d, e

  such that the monad laws hold:

$$f^\dagger_1 \circ \eta_X = f \qquad (\eta_X)^\dagger_d = \mathrm{id}_{TXd} \qquad (g^\dagger_e \circ f)^\dagger_d = g^\dagger_{d \cdot e} \circ f^\dagger_d$$

Example: the graded monad List has

- graded set $\mathrm{List}X$ for each set X
- singleton functions $X \to \mathrm{List}X1$
- $f^\dagger_d[x_1, \ldots, x_k] = f x_1 + \!\!\!+ \cdots + \!\!\!+ f x_k$

Graded monads

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- A T-*algebra* $\mathsf{A} = (A, (-)^\ddagger)$ is a graded set $U_\mathsf{T}\mathsf{A} = A$, with an extension operation

$(\mathcal{C}, U)$ is *graded monadic* if $(\mathcal{C}, U) \cong (\mathbf{Alg}(\mathsf{T}), U_\mathsf{T})$ for some (unique) graded monad T

The problem with graded monads

Graded monoids are not graded monadic, hence do not have a rigidly graded presentation

There is a concrete functor

$$\begin{array}{ccc} \mathbf{GMon} & \xrightarrow{R} & \mathbf{Alg}(\mathsf{List}) \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle U_{\mathsf{List}}} & \\ & \mathbf{GSet} & \end{array}$$

satisfying a universal property, but it is not an isomorphism

Similarly for graded rings, . . .
(but graded modules have a rigidly graded presentation)

Graded presentations are too rigid

Each operation $\mathrm{op} \in \Sigma(n, d)$ is interpreted as

$$[\![\mathrm{op}]\!]_e : (Ae)^n \longrightarrow A(d \cdot e)$$

but we want

$$\mathsf{m}_{d_1,d_2} : Ad_1 \times Ad_2 \longrightarrow A(d_1 + d_2)$$

This work

Develop a notion of flexibly graded presentation

- Every flexibly graded presentation (Σ, E) induces
  - a canonical graded monad $\mathsf{T}_{(\Sigma,E)}$
  - along with a flexibly graded algebraic operation for each operation of the presentation
- Examples like List have computationally natural flexibly graded presentations
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Flexibly graded presentations

A flexibly graded presentation (Σ, E) consists of

- a signature Σ: sets

  $$\Sigma(d'_1, \ldots, d'_n; d)$$

  of operations

  $$\frac{e \in \mathbb{N} \quad \Gamma \vdash t_1 : d'_1 \cdot e \quad \cdots \quad \Gamma \vdash t_n : d'_n \cdot e}{\Gamma \vdash \mathrm{op}(e; t_1, \ldots, t_n) : d \cdot e}$$

  such as $\mathsf{m}_{d_1,d_2} \in E(d_1, d_2; (d_1 + d_2))$
- a collection of axioms E: sets

  $$E(d'_1, \ldots, d'_n; d)$$

  of equations

  $$x_1 : d'_1, \ldots, x_n : d'_n \vdash t \equiv u : d$$

  such as

  $$\mathsf{m}_{d_1+d_2,d_3}(1; \mathsf{m}_{d_1,d_2}(1; x, y), z) \equiv \mathsf{m}_{d_1,d_2+d_3}(1; x, \mathsf{m}_{d_2,d_3}(1; y, z))$$

Semantics

For every flexibly graded presentation (Σ, E), there is:

- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$

A (Σ, E)-algebra $(A, [\![-]\!])$ is:

- a graded set A
- with an interpretation

$$[\![\mathsf{op}]\!]_e : \prod_i A(d'_i \cdot e) \to A(d \cdot e) \qquad \text{natural in } e$$

  of each $\mathsf{op} \in \Sigma(d'_1, \ldots, d'_n;\, d)$
- satisfying each axiom $t \equiv u$ of E:

$$[\![t]\!]_e = [\![u]\!]_e \qquad \text{for every } e$$

Semantics

For every flexibly graded presentation (Σ, E), there is:

- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$
- a sound and complete equational logic

$\Gamma \vdash t \equiv u : d$ generated by

$$\frac{(t, u) \in E(d'_1, \ldots, d'_n; d) \qquad \Gamma \vdash s_1 : d'_1 \cdot e \qquad \cdots \qquad \Gamma \vdash s_n : d'_n \cdot e}{\Gamma \vdash t\{e; x_1 \mapsto s_1, \ldots, x_n \mapsto s_n\} \equiv u\{e; x_1 \mapsto s_1, \ldots, x_n \mapsto s_n\} : d \cdot e}$$

and some other rules

Soundness and completeness:

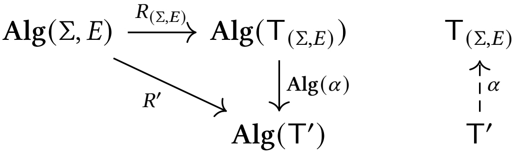
$$[\![t]\!] = [\![u]\!] \text{ in every } (\Sigma, E)\text{-algebra} \quad \Leftrightarrow \quad \Gamma \vdash t \equiv u : d \text{ is derivable}$$

Semantics

For every flexibly graded presentation (Σ, E), there is:

- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$
- a sound and complete equational logic
- a graded monad $\mathsf{T}_{(\Sigma,E)}$ on $\mathbf{Set}$ and concrete functor $R_{(\Sigma,E)} : \mathbf{Alg}(\Sigma, E) \to \mathbf{Alg}(\mathsf{T}_{(\Sigma,E)})$, with a universal property

For every graded monad T' and concrete functor $R' : \mathbf{Alg}(\Sigma, E) \to \mathbf{Alg}(\mathsf{T}')$:



(But $R_{(\Sigma,E)}$ is usually not an isomorphism)

Semantics

For every flexibly graded presentation (Σ, E), there is:

- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$
- a sound and complete equational logic
- a graded monad $\mathsf{T}_{(\Sigma,E)}$ on $\mathbf{Set}$ and concrete functor $R_{(\Sigma,E)} : \mathbf{Alg}(\Sigma, E) \to \mathbf{Alg}(\mathsf{T}_{(\Sigma,E)})$, with a universal property
- for every op in Σ, a flexibly graded algebraic operation

For op $\in \Sigma(d'_1, \ldots, d'_n;\, d)$:

$$\alpha_{\mathsf{op},X,e} : \prod_i T_{(\Sigma,E)}X(d'_i \cdot e) \to T_{(\Sigma,E)}X(d \cdot e)$$

natural in e, and compatible with Kleisli extension

(Because each free $\mathsf{T}_{(\Sigma,E)}$-algebra $T_{(\Sigma,E)}X$ forms a (Σ, E)-algebra)

Semantics

For every flexibly graded presentation (Σ, E), there is:

- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$
- a sound and complete equational logic
- a graded monad $\mathsf{T}_{(\Sigma,E)}$ on $\mathbf{Set}$ and concrete functor $R_{(\Sigma,E)} : \mathbf{Alg}(\Sigma, E) \to \mathbf{Alg}(\mathsf{T}_{(\Sigma,E)})$, with a universal property
- for every op in Σ, a flexibly graded algebraic operation

A large class of graded monads have flexibly graded presentations:

- exactly the finitary graded monads on $\mathbf{Set}$
- correspondence goes via flexibly graded clones

Graded monads we care about have natural flexibly graded presentations

Summary

Given a flexibly graded presentation (Σ, E), there is

- a graded monad $T_{(\Sigma,E)}$
- with a $(d'_1, \ldots, d'_n; d)$-ary algebraic operation

$$[\![\mathrm{op}]\!]_{X,e} : \prod_i T_{(\Sigma,E)} X(d'_i \cdot e) \to T_{(\Sigma,E)} X(d \cdot e)$$

  for each $\mathrm{op} \in \Sigma(d'_1, \ldots, d'_n; d)$ (satisfying equations)
- that is in some sense canonical, even if it does not quite capture (Σ, E)-algebras

Details are in the papers:

- Dylan McDermott and Tarmo Uustalu, Flexibly graded monads and graded algebras, MPC 2022
- Shin-ya Katsumata, Dylan McDermott, Tarmo Uustalu and Nicolas Wu, Flexible presentations of graded monads, ICFP 2022

Constructing $T_{(\Sigma,E)}$

flexibly graded presentations

$\downarrow$ $(\Sigma,E) \mapsto \mathrm{Tm}_{(\Sigma,E)}$

flexibly graded clones $\quad = $ sets of terms, with variables and sub

$\downarrow$ left Kan extension along $\mathbf{FCtx} \to \mathbf{GSet}$

flexibly graded monads $\quad = $ monad on $\mathbf{GSet}$

$\downarrow$ compose with $K : \mathbf{RSet} \to \mathbf{GSet}$

graded monads

Constructing $T_{(\Sigma,E)}$

algebraic theories and relative monads are closely connected (jww Nathanael Arkor)

flexibly graded presentations

$(\Sigma,E) \mapsto \mathrm{Tm}_{(\Sigma,E)}$ ↓ $\simeq$ ↑

flexibly graded clones = $(\mathbf{FCtx} \to \mathbf{GSet})$-relative monad

left Kan extension along $\mathbf{FCtx} \to \mathbf{GSet}$ ↓ $\simeq$ ↑ compose with $\mathbf{FCtx} \to \mathbf{GSet}$

flexibly graded monads preserving conical sifted colimits = monad on $\mathbf{GSet}$ preserving conical sifted colimits

↑ $\dashv$ ↓ compose with $K\colon \mathbf{RSet} \to \mathbf{GSet}$

graded monads preserving conical sifted colimits = K-relative monad preserving conical sifted colimits

Monads as models of computational effects

Let T be the monad that arises from a presentation (Σ, E). Then:

- an element $t \in TX$ can be thought of as a computation over X
- the unit functions $\eta_X : X \to TX$ provide trivial computations
- the Kleisli extension functions $(X \to TY) \to (TX \to TY)$ provide sequencing of computations
- the interpretation functions

$$[\![\mathrm{op}]\!] : (TX)^n \to TX \qquad \text{where } (\mathrm{op} : n) \in \Sigma$$

  provide effectful operations

Example: if (Σ, E) is the presentation of monoids, then

- a computation $t \in TX = \mathrm{List}X$ is a list of alternatives;
- $TX = \mathrm{List}X$ is a monoid, with

  unit $[\![\mathsf{u}]\!] : 1 \to \mathrm{List}X$ the empty list

  multiplication $[\![\mathsf{m}]\!] : \mathrm{List}X \times \mathrm{List}X \to \mathrm{List}X$ concatenation of lists