Sweedler Theory of Monads

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Abstract. Monad-comonad interaction laws are a mathematical concept for describing communication protocols between effectful computations and coeffectful environments in the paradigm where notions of effectful computation are modelled by monads and notions of coeffectful environment by comonads. We show that monad-comonad interaction laws are an instance of measuring maps from Sweedler theory for duoidal categories whereby the final interacting comonad for a monad and a residual monad arises as the Sweedler hom and the initial residual monad for a monad and an interacting comonad as the Sweedler copower. We then combine this with a (co)algebraic characterization of monad-comonad interaction laws to derive descriptions of the Sweedler hom and the Sweedler copower in terms of their coalgebras resp. algebras.

Keywords: (co)monads · (co)algebras · interaction laws · runners · duoidal categories · Sweedler operations

1 Introduction

The monad-comonad interaction laws of Katsumata et al. [16] are a mathematical concept for formalizing ways in which effectful programs (e.g., programs reading from and writing to a store, programs making nondeterministic choices) can be run. The idea is that effectful programs issue requests to the outside world; they can thus run on machines that can service such requests. Programs denote computations, machines implement environments. Notions of computation are modelled by monads in the manner first explained by Moggi [23], while notions of environment can be modelled by comonads. Interaction laws model protocols of cooperation between computations and environments. Ideally, interaction should result in a return value and a final state. But it may be that some effects cannot be serviced, in which case interaction yields a residual computation of a return value and a final state; another monad is then needed to model the suitable notion of residual computation. A monad-comonad interaction law is therefore given by a monad T, a comonad D and a monad R on a symmetric monoidal category with a family of maps $TX \otimes DY \to R(X \otimes Y)$ natural in X and Y and agreeing with the (co)units and (co)multiplications. If $R = \mathsf{Id}$, we have a non-residual interaction law.

It is natural to ask for useful methods for recognizing and constructing monad-comonad interaction laws. Specifically, it would be useful to find: a final monad for a given interacting comonad and residual monad; a final interacting comonad for a given monad and residual monad; or an initial residual monad for a given monad and interacting comonad.

In this paper, we show how to find these universal (co)monads, elaborating on some ideas and results from prior work on interaction [16,33]. We emphasize that the most important structural foundation for interaction laws is the duoidal [10,2] interrelationship of the composition and Day convolution monoidal structures in endofunctor categories. It is so significant that some central statements about interaction laws can be made on the level of monoids and comonoids in general symmetric closed duoidal categories, completely suppressing any specifics about monads and comonads. In fact, it turns out that monad-comonad interaction laws are an instance of measuring maps from the Sweedler theory for duoidal categories as developed by López Franco and Vasilakopoulou [20]. The universal (co)monads are instances of the operations studied in this theory. In particular, the final interacting comonad is an instance of the Sweedler hom and the initial residual monad is an instance of the Sweedler copower.

To obtain results about monad-comonad interaction specifically, we combine this general perspective with the characterization of monad-comonad interaction laws by Uustalu and Voorneveld [33] as functors between the categories of (co)algebras of the (co)monads involved. This allows us to describe the Sweedler hom and the Sweedler power via their categories of (co)algebras in terms of what we call stateful and continuation-based runners.

We also discuss an enriched version of monad-comonad interaction laws, of which strong monad-comonad interaction laws are a special case. In this case, both kinds of runners of an enriched monad on a self-enriched category can be viewed as its algebras in another enriched category.

The paper is organized as follows. First, in Sect. 2, we review the basics of monad-comonad interaction laws. In Sect. 3, we show that monad-comonad interaction laws, the universal interacting comonad and the universal residual monad are an instance of measuring maps, the Sweedler hom and the Sweedler copower in symmetric closed duoidal categories. We then review the (co)algebraic perspective on monad-comonad interaction laws in Sect. 4, and apply it to derive (co)algebraic characterizations of the Sweedler hom and the Sweedler copower in Sect. 5. In Sect. 6, we comment on enriched monad-comonad interaction laws. We review some background category theory literature and related semantics work in Sect. 7. New material is primarily in Sects. 5, 6; some statements in Sect. 4 are also new.

We assume from the reader familiarity with the use of (strong) monads in mathematical semantics to model notions of effectful computation, and familiarity with the basics of the categorical machinery we need (monads and comonads, symmetric monoidal closed categories, accessibility [21,1], enrichment [17]).

2 Monad-Comonad Interaction Laws

We begin by reviewing the basics of monad-comonad interaction laws [16].

Consider a symmetric monoidal closed category $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$, e.g., a Cartesian monoidal closed category, e.g., **Set**.

A *(residual) functor-functor interaction law* is given by endofunctors F, G, H on $\mathbb{C}$ together with a family of maps

$$\phi_{X,Y} : FX \otimes GY \to H(X \otimes Y)$$

natural in X, Y. We speak of a non-residual interaction law when $H = \mathsf{Id}$. A *map between (residual) functor-functor interaction laws* (F, G, H, ϕ) and (F', G', H', ϕ') is given by natural transformations $f : F \to F'$, $g : G' \to G$ and $h : H \to H'$ satisfying the equation

$$\begin{tikzcd}
 & FX \otimes GY \arrow[r, "\psi_{X,Y}"] & H(X \otimes Y) \arrow[dd, "h_{X \otimes Y}"] \\
FX \otimes G'Y \arrow[ur, "FX \otimes g_Y"] \arrow[dr, "f_X \otimes G'Y"'] & & \\
 & F'X \otimes G'Y \arrow[r, "\psi'_{X,Y}"] & H'(X \otimes Y)
\end{tikzcd}$$

Functor-functor interaction laws form a category that has a monoidal structure based on endofunctor composition.

A *(residual) monad-comonad interaction law* is given by a monad T, a comonad D and a monad R on $\mathbb{C}$ with a family of maps

$$\psi_{X,Y} : TX \otimes DY \to R(X \otimes Y)$$

natural in X, Y, that additionally satisfies the equations

$$\begin{tikzcd}
 & X \otimes Y \arrow[r, equal] & X \otimes Y \arrow[dd, "\eta^R_{X,Y}"] \\
X \otimes DY \arrow[ur, "\mathsf{id} \otimes \varepsilon_Y"] \arrow[dr, "\eta_X \otimes \mathsf{id}"'] & & \\
 & TX \otimes DY \arrow[r, "\psi_{X,Y}"] & R(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TTX \otimes DDY \arrow[r, "\psi_{TX,DY}"] & R(TX \otimes DY) \arrow[r, "R\psi_{X,Y}"] & RR(X \otimes Y) \arrow[dd, "\mu^R_{X,Y}"] \\
TTX \otimes DY \arrow[ur, "\mathsf{id} \otimes \delta_Y"] \arrow[dr, "\mu_X \otimes \mathsf{id}"'] & & & \\
 & TX \otimes DY \arrow[rr, "\psi_{X,Y}"] & & R(X \otimes Y)
\end{tikzcd}$$

(Every such interaction law gives a functor-functor interaction law (UT, UD, UR, ψ), where U sends (co)monads to their underlying functors.) When $R = \mathsf{Id}$, we speak of a non-residual interaction law. A *map between (residual) monad-comonad interaction laws* (T, D, R, ψ) and (T', D', R', ψ') is given by a monad map $T \to T'$, a comonad map $D' \to D$ and a monad map $R \to R'$ that make a map between the underlying functor-functor interaction laws. Monad-comonad interaction laws form a category isomorphic to the category of monoid objects in the category of functor-functor interaction laws.

Example 1. Let $\mathbb{C} = \mathbf{Set}$ (or any SMCC). Take $TX = S \Rightarrow (S \times X)$ (the state monad) and $DX = S_0 \times (S_0 \Rightarrow X)$ (the costate monad). There is a non-residual monad-comonad interaction law of T, D when $S = S_0$ and more generally when S, S_0 come with a lens structure $\mathit{get} : S_0 \to S$, $\mathit{put} : S_0 \times S \to S_0$; in fact, these laws are in bijection with lenses.

Let $\mathbb{C} = \mathbf{Set}$ (or any extensive category that also has the relevant initial algebras and final coalgebras). Take $FX = 1 + X^2$ and T the free monad on

F, so $TX \cong \mu X'.X + 1 + X'^2$ (leaf-labelled nullary-binary trees). The only comonad D that can interact with T non-residually is $DY \cong 0$. If we take $RZ = 1 + Z$, we have an R-residual interaction law of T and D for example for $DY \cong \nu Y'.Y \times (2 \times Y')$ (node-labelled bitstreams), i.e., the cofree comonad for $GY = 2 \times Y$.

See [16,33] for further examples and their intuitive meaning for semantics.

Some equivalent formulations of interaction laws will be useful. Due to the bijections

$$\begin{array}{c} FX \otimes GY \to H(X \otimes Y) \text{ nat. in } X,\, Y \\ \hline\hline \mathbb{C}(X \otimes Y, Z) \to \mathbb{C}(FX \otimes GY, HZ) \text{ nat. in } X,\, Y,\, Z \\ \hline\hline \mathbb{C}(X, Y \multimap Z) \to \mathbb{C}(FX, GY \multimap HZ) \text{ nat. in } X,\, Y,\, Z \\ \hline\hline F(Y \multimap Z) \to GY \multimap HZ \text{ nat. in } Y,\, Z \end{array}$$

an H-residual functor-functor interaction law of F, G is the same as a family of maps

$$\phi_{Y,Z} : F(Y \multimap Z) \to GY \multimap HZ$$

natural in Y, Z. Under this view, the equation required of a functor-functor interaction law map (f, g, h) between (F, G, H, ϕ) and (F', G', H', ϕ') becomes

$$\begin{tikzcd} F(Y \multimap Z) \arrow[r, "\phi_{Y,Z}"] \arrow[d, "f_{Y \multimap Z}"'] & GY \multimap HZ \arrow[d, "g_Y \multimap h_Z"] \\ F'(Y \multimap Z) \arrow[r, "\phi'_{Y,Z}"] & G'Y \multimap H'Z \end{tikzcd}$$

An R-residual monad-comonad interaction law of T, D is the same as a family of maps

$$\psi_{Y,Z} : T(Y \multimap Z) \to DY \multimap RZ$$

natural in Y, Z satisfying

$$\begin{tikzcd} Y \multimap Z \arrow[r, equal] \arrow[d, "\eta_{Y \multimap Z}"'] & Y \multimap Z \arrow[d, "\varepsilon_Y \multimap \eta^R_Z"] \\ T(Y \multimap Z) \arrow[r, "\psi_{Y,Z}"] & DY \multimap RZ \end{tikzcd} \qquad \begin{tikzcd} TT(Y \multimap Z) \arrow[r, "T\psi_{Y,Z}"] \arrow[d, "\mu_{Y \multimap Z}"'] & T(DY \multimap RZ) \arrow[r, "\psi_{DY,RZ}"] & DDY \multimap RRZ \arrow[d, "\delta_Y \multimap \mu^R_Z"] \\ T(Y \multimap Z) \arrow[rr, "\psi_{Y,Z}"] & & DY \multimap RZ \end{tikzcd}$$

Suppose $F, G, H : \mathbb{C} \to \mathbb{C}$ are such that the coends and ends

$$\begin{array}{rl} (F \star G)\, Z = \int^{X,Y} \mathbb{C}(X \otimes Y, Z) \bullet (FX \otimes GY) & = \int^{Y} F(Y \multimap Z) \otimes GY \\ (G \mathrel{-\!\star} H)\, X = \int_{Y,Z} \mathbb{C}(X, Y \multimap Z) \pitchfork (GY \multimap HZ) & = \int_{Y} GY \multimap H(X \otimes Y) \end{array}$$

exist. ($F \star G$ is called the *Day convolution*.) Then, because of the bijections

$$\begin{array}{c} \int^{X,Y} \mathbb{C}(X \otimes Y, Z) \bullet (FX \otimes GY) \to HZ \text{ nat. in } Z \\ \hline\hline \mathbb{C}(X \otimes Y, Z) \to \mathbb{C}(FX \otimes GY, HZ) \text{ nat. in } X,\, Y,\, Z \\ \hline\hline \mathbb{C}(X, Y \multimap Z) \to \mathbb{C}(FX, GY \multimap HZ) \text{ nat. in } X,\, Y,\, Z \\ \hline\hline FX \to \int_{Y,Z} \mathbb{C}(X, Y \multimap Z) \pitchfork (GY \multimap HZ) \text{ nat. in } X \end{array}$$

an H-residual functor-functor interaction law of F, G turns out to be the same as a natural transformation $F \star G \to H$ or $F \to G \multimap\!\!\star\, H$. An R-residual monad-comonad interaction law of T, D is the same as a natural transformation $UT \star UD \to UR$ satisfying certain equations and also—by way of a particularly concise characterization—the same as a monad map $T \to D \multimap\!\!\star\, R$ where $D \multimap\!\!\star\, R$ is a certain canonical monad with $UD \multimap\!\!\star\, UR$ as the underlying functor.

Now, if $\mathbb{C}$ is locally presentable and F, G, H are accessible, then $F \star G$ and $G \multimap\!\!\star\, H$ are guaranteed to exist and be accessible. Writing $[\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$ for the category of accessible endofunctors on $\mathbb{C}$, we obtain functors $\star : [\mathbb{C}, \mathbb{C}]_{\mathrm{a}} \times [\mathbb{C}, \mathbb{C}]_{\mathrm{a}} \to [\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$ and $\multimap\!\!\star : [\mathbb{C}, \mathbb{C}]_{\mathrm{a}}^{\mathrm{op}} \times [\mathbb{C}, \mathbb{C}]_{\mathrm{a}} \to [\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$. Together with $J \in [\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$ defined by $JZ = \mathbb{C}(\mathsf{I}, Z) \bullet \mathsf{I}$, the functor $\star$ equips $[\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$ with a symmetric monoidal structure. We also get that $- \star G \vdash G \multimap\!\!\star\, -$, i.e., this structure is closed.[3] The functor $\multimap\!\!\star : [\mathbb{C}, \mathbb{C}]_{\mathrm{a}}^{\mathrm{op}} \times [\mathbb{C}, \mathbb{C}]_{\mathrm{a}} \to [\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$ is lax monoidal wrt. the composition monoidal structure on $[\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$. That $UD \multimap\!\!\star\, UR$ carries a monad structure if D is an accessible comonad and R is an accessible monad is a consequence of this.

These observations suggest the possibility of abstraction by switching to a more general setting. Instead of considering $[\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$, we can consider an arbitrary category $\mathbb{D}$ equipped with a monoidal structure and a symmetric monoidal structure that suitably agree. The appropriate notion of agreement is duoidality [10,2]. We will next consider this abstraction and see that monad-comonad interaction laws are the measuring maps of an instance of López Franco and Vasilakopoulou's Sweedler theory for duoidal categories [20].

3 Sweedler Theory for Duoidal Categories

We review the Sweedler theory for duoidal categories [20] and show that monads provide an instance.

Assume a symmetric duoidal category $(\mathbb{D}, I, \diamond, J, \star)$, i.e., a symmetric monoidal category in $\mathbf{MonCAT}_{\mathrm{oplax}}$, that is also closed in the sense that $- \star G$ has a right adjoint $G \multimap\!\!\star\, -$ in $\mathbf{CAT}$. Explicitly, this means that we have a category $\mathbb{D}$ equipped with a monoidal structure $(I, \diamond)$, a symmetric monoidal closed structure $(J, \star, \multimap\!\!\star)$ and structural laws

$$
\begin{array}{cc}
J \to I & J \to J \diamond J \\
I \star I \to I & (F \diamond G) \star (H \diamond K) \to (F \star H) \diamond (G \star K)
\end{array}
$$

satisfying appropriate equations witnessing oplax monoidality of $J : \mathbf{1} \to \mathbb{D}$ and $\star : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ as functors between monoidal categories for the $(I, \diamond)$ monoidal structure on $\mathbb{D}$.

[3] If $\mathbb{C}$ is locally κ-presentable with the κ-presentable objects closed under I and $\otimes$, then the κ-accessible endofunctors on $\mathbb{C}$ form a monoidal category with $\star$ as tensor. Garner and López Franco [13, Sect. 8.1] show that this monoidal category is closed, but their closed structure is different from ours. Our $G \multimap\!\!\star\, H$ has the property that natural transformations $F \to G \multimap\!\!\star\, H$ are H-residual functor-functor interaction laws of F, G even if F is not accessible; this is not the case for Garner and López Franco's. This is why we do not restrict to fixed κ, and instead use all of $[\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$.

The internal hom object $F \mathrel{-\!\!\star} I$ is called the *dual* of F. Stretching this terminology, the object $F \mathrel{-\!\!\star} H$ can be called the dual of F wrt. H.

We write $\mathbf{Mon}(\mathbb{D})$ (respectively $\mathbf{Comon}(\mathbb{D})$) for the categories of monoids (resp. comonoids) in $\mathbb{D}$ wrt. the $(I, \diamond)$ monoidal structure.

The composition monoidal and Day convolution symmetric monoidal closed structures $(\mathsf{Id}, \cdot)$ and $(J, \star, \mathrel{-\!\!\star})$ on $[\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$ yield an example of such a symmetric duoidal category $\mathbb{D}$. The categories $\mathbf{Mon}([\mathbb{C}, \mathbb{C}]_{\mathrm{a}})$ and $\mathbf{Comon}([\mathbb{C}, \mathbb{C}]_{\mathrm{a}})$ are those of accessible monads and comonads.

The object J has a comonoid structure $J \to I$, $J \to J \diamond J$, and the functor $\mathrel{-\!\!\star} : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{D}$ is lax monoidal wrt. the $(I, \diamond)$ monoidal structure. The operations

$$\begin{aligned} \star &: \mathbb{D} \times \mathbb{D} \to \mathbb{D} \\ \mathrel{-\!\!\star} &: \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{D} \end{aligned}$$

lift to

$$\begin{array}{rll} \star : \mathbf{Comon}(\mathbb{D}) \times \mathbf{Comon}(\mathbb{D}) \to \mathbf{Comon}(\mathbb{D}) & \text{tensor of comonoids} \\ \mathrel{-\!\!\star} : (\mathbf{Comon}(\mathbb{D}))^{\mathrm{op}} \times \mathbf{Mon}(\mathbb{D}) \to \mathbf{Mon}(\mathbb{D}) & \text{power of a monoid} \end{array}$$

in the sense that

$$\begin{tikzcd} \mathbf{Comon}(\mathbb{D}) \times \mathbf{Comon}(\mathbb{D}) \arrow[r, "\star"] \arrow[d, "U \times U"'] & \mathbf{Comon}(\mathbb{D}) \arrow[d, "U"] \\ \mathbb{D} \times \mathbb{D} \arrow[r, "\star"] & \mathbb{D} \end{tikzcd} \qquad \begin{tikzcd} (\mathbf{Comon}(\mathbb{D}))^{\mathrm{op}} \times \mathbf{Mon}(\mathbb{D}) \arrow[r, "\mathrel{-\!\!\star}"] \arrow[d, "U^{\mathrm{op}} \times U"'] & \mathbf{Mon}(\mathbb{D}) \arrow[d, "U"] \\ \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \arrow[r, "\mathrel{-\!\!\star}"] & \mathbb{D} \end{tikzcd}$$

via

$$\varepsilon = \ D_0 \star D_1 \xrightarrow{\varepsilon_0 \star \varepsilon_1} I \star I \longrightarrow I$$

$$\delta = \ D_0 \star D_1 \xrightarrow{\delta_0 \star \delta_1} (D_0 \diamond D_0) \star (D_1 \diamond D_1) \longrightarrow (D_0 \star D_1) \diamond (D_0 \star D_1)$$

$$\eta = \ I \longrightarrow I \mathrel{-\!\!\star} I \xrightarrow{\varepsilon \mathrel{-\!\!\star} \eta^R} D \mathrel{-\!\!\star} R$$

$$\mu = \ (D \mathrel{-\!\!\star} R) \diamond (D \mathrel{-\!\!\star} R) \longrightarrow (D \diamond D) \mathrel{-\!\!\star} (R \diamond R) \xrightarrow{\delta \mathrel{-\!\!\star} \mu^R} D \mathrel{-\!\!\star} R$$

Comonoid maps $D_0 \star D_1 \to D$ are the same as maps $\psi : UD_0 \star UD_1 \to UD$ satisfying

$$\begin{tikzcd} D_0 \star D_1 \arrow[r, "\psi"] \arrow[d, "\varepsilon_0 \star \varepsilon_1"'] & D \arrow[d, "\varepsilon"] \\ I \star I \arrow[r] & I \end{tikzcd} \qquad \begin{tikzcd} D_0 \star D_1 \arrow[rr, "\psi"] \arrow[d, "\delta_0 \star \delta_1"'] & & D \arrow[d, "\delta"] \\ (D_0 \diamond D_0) \star (D_1 \diamond D_1) \arrow[r] & (D_0 \star D_1) \diamond (D_0 \star D_1) \arrow[r, "\psi \diamond \psi"] & D \diamond D \end{tikzcd}$$

(omitting the Us in the equations). Such maps ψ could be called D-residual comonoid-comonoid interaction laws of D_0, D_1.

Monoid maps $T \to D \mathrel{-\!\!\star} R$ are in bijection with maps $\psi : UT \star UD \to UR$ that satisfy

$$\begin{tikzcd} & I \star I \arrow[r] & I \arrow[d, "\eta^R"] \\ I \star D \arrow[ur, "I \star \varepsilon"] \arrow[dr, "\eta \star D"'] & & \\ & T \star D \arrow[r, "\psi"] & R \end{tikzcd} \qquad \begin{tikzcd} & (T \diamond T) \star (D \diamond D) \arrow[r] & (T \star D) \diamond (T \star D) \arrow[r, "\psi \diamond \psi"] & R \diamond R \arrow[d, "\mu^R"] \\ (T \diamond T) \star D \arrow[ur, "(T \diamond T) \star \delta"] \arrow[dr, "\mu \star D"'] & & & \\ & T \star D \arrow[rr, "\psi"] & & R \end{tikzcd}$$

(again omitting the Us in the equations), which are known as *measuring maps* from T to R by D and which we can also call R-residual monoid-comonoid interaction laws of T, R.

The three *Sweedler operations*

$\mathcal{C} : (\mathbf{Comon}(\mathbb{D}))^{\mathrm{op}} \times \mathbf{Comon}(\mathbb{D}) \to \mathbf{Comon}(\mathbb{D})$	internal hom of comonoids
$\rhd : \mathbf{Comon}(\mathbb{D}) \times \mathbf{Mon}(\mathbb{D}) \to \mathbf{Mon}(\mathbb{D})$	Sweedler copower of a monoid
$\mathcal{M} : (\mathbf{Mon}(\mathbb{D}))^{\mathrm{op}} \times \mathbf{Mon}(\mathbb{D}) \to \mathbf{Comon}(\mathbb{D})$	Sweedler hom of monoids (univ. measuring comonoid)

are everywhere defined by the following adjunctions if the adjoints exist.

$$
\begin{tikzcd}
\mathbf{Comon}(\mathbb{D}) \arrow[d, bend right, "-\star D_1"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{Comon}(\mathbb{D}) \arrow[u, bend right, "\mathcal{C}(D_1,-)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Mon}(\mathbb{D}) \arrow[d, bend right, "D\rhd -"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{Mon}(\mathbb{D}) \arrow[u, bend right, "D \multimap\!\star -"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Comon}(\mathbb{D}) \arrow[d, bend right, "-\rhd T"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{Mon}(\mathbb{D}) \arrow[u, bend right, "\mathcal{M}(T,-)"']
\end{tikzcd}
$$

They are defined for specific pairs of (co)monoids if the universal objects specified by the following bijections exist.

$$
\begin{array}{c}
D_0 \star D_1 \to D \\ \hline\hline
D_0 \to \mathcal{C}(D_1, D)
\end{array}
\qquad\qquad
\begin{array}{c}
UT \star UD \to UR \text{ meas.} \\ \hline\hline
T \to D \multimap\!\star R \\ \hline\hline
D \rhd T \to R \\ \hline\hline
D \to \mathcal{M}(T, R)
\end{array}
$$

The comonoid $\mathcal{M}(T, I)$ is called the *Sweedler dual* of the monoid T.

By definition, the comonoid $\mathcal{C}(D_1, D)$ is the final comonoid interacting with the comonoid D_1 D-residually. The Sweedler hom $\mathcal{M}(T, R)$ is the final R-residually interacting comonoid for the monoid T. The Sweedler copower $D \rhd T$ is the initial residual monoid for monoid-comonoid interactions of T and D.

If the Sweedler operations are everywhere defined, for which it suffices that $\mathbb{D}$ is locally presentable [20, Thm. 20], then the category $(\mathbf{Comon}(\mathbb{D}), J, \star, \mathcal{C})$ is symmetric monoidal closed and the category $(\mathbf{Mon}(\mathbb{D}), \rhd, \multimap\!\star, \mathcal{M})$ is copowered, powered and enriched over $(\mathbf{Comon}(\mathbb{D}), J, \star, \mathcal{C})$. However, local presentability of $\mathbb{C}$ is not enough for local presentability (or even accessibility) of $[\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$ (for example, $[\mathbf{Set}, \mathbf{Set}]_{\mathrm{a}}$ is not accessible). In Sect. 5, we return to the question of everywhere-definedness of the Sweedler operations for $[\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$.

The Sweedler theory perspective allows us to establish some facts about interaction laws of free monads very easily. For example, we can straightforwardly derive a characterization of measuring maps from the free monoid F^* on F (assuming it exists).

Proposition 1. *Measuring maps $U(F^*) \star UD \to UR$ are in bijection with maps $F \star UD \to UR$.*

Proof. This is witnessed by the following chain of bijections.

$$
\begin{array}{c}
F \star UD \to UR \\ \hline\hline
F \to UD \multimap\!\star UR \\ \hline\hline
F \to U(D \multimap\!\star R) \\ \hline\hline
F^* \to D \multimap\!\star R \\ \hline\hline
U(F^*) \star UD \to UR \text{ meas.}
\end{array}
$$

$\square$

Similarly, we can calculate closed-form expressions for the Sweedler hom from a free monoid and the Sweedler copower of a free monoid. Here $G^\dagger$ denotes the cofree comonoid on G (if it exists).

Proposition 2. (i) $\mathcal{M}(F^*, R) \cong (F \mathrel{-\!\!\star} UR)^\dagger$. (ii) $D \rhd F^* \cong (F \star UD)^*$.

Proof. (i) As witnessed by the chain of bijections on the left below, comonoid maps $D \to \mathcal{M}(F^*, R)$ and comonoid maps $D \to (F \mathrel{-\!\!\star} UR)^\dagger$ are in bijection naturally in D. (ii) The chain of bijections on the right below composes to a bijection natural in R between monoid maps $D \rhd F^* \to R$ and monoid maps $(F \star UD)^* \to R$.

$$
\begin{array}{c}
D \to (F \mathrel{-\!\!\star} UR)^\dagger \\ \hline\hline
UD \to F \mathrel{-\!\!\star} UR \\ \hline\hline
F \to UD \mathrel{-\!\!\star} UR \\ \hline\hline
F \to U(D \mathrel{-\!\!\star} R) \\ \hline\hline
F^* \to D \mathrel{-\!\!\star} R \\ \hline\hline
D \to \mathcal{M}(F^*, R)
\end{array}
\qquad\qquad
\begin{array}{c}
(F \star UD)^* \to R \\ \hline\hline
F \star UD \to UR \\ \hline\hline
F \to UD \mathrel{-\!\!\star} UR \\ \hline\hline
F \to U(D \mathrel{-\!\!\star} R) \\ \hline\hline
F^* \to D \mathrel{-\!\!\star} R \\ \hline\hline
D \rhd F^* \to R
\end{array}
$$

□

Example 2. Let $\mathbb{C} = \mathbf{Set}$. (i) Take $F = 0$, then $F^* \cong \mathsf{Id}$. We can calculate $F \mathrel{-\!\!\star} UR \cong 1$, therefore $\mathcal{M}(F^*, R) \cong \mathsf{Id}$, for any monad R.

Next take $FX = X^2$, then $F^*X \cong \mu X'.\, X + X'^2$ (these are leaf-labelled binary trees). We can calculate $(F \mathrel{-\!\!\star} UR)Y \cong R\,(2 \times Y)$, hence $\mathcal{M}(F^*, R)\,Y \cong \nu Y'.\, Y \times R\,(2 \times Y')$ (node-labelled streams of bits for $R = \mathsf{Id}$, node-labelled nonempty colists of bits for $RZ = 1 + Z$).

Finally, take $FX = 1 + X^2$, then $F^*X \cong \mu X'.\, X + 1 + X'^2$ (leaf-labelled nullary-binary trees). We calculate $(F \mathrel{-\!\!\star} UR)Y \cong R\,0 \times R\,(2 \times Y)$, hence $\mathcal{M}(F^*, R)\,Y \cong \nu Y'.\, Y \times R\,0 \times R\,(2 \times Y')$. For $R = \mathsf{Id}$ and any R such that $R\,0 \cong 0$, this means that $\mathcal{M}(F^*, R) \cong 0$. For $RZ = 1 + Z$, we get $\mathcal{M}(F^*, R)\,Y \cong \nu Y'.\, Y \times (1 + 2 \times Y')$ (node-labelled nonempty colists of bits).

(ii) Take $F = 0$, then $F^\star \cong \mathsf{Id}$. We can calculate $(F \star UD) \cong 0$, hence $D \rhd F^* \cong \mathsf{Id}$, for any comonad D.

Take $FX = X^2$, then $F^*X \cong \mu X'.\, X + X'^2$. We can calculate $(F \star UD)\,Z \cong D\,(Z^2)$, therefore $(D \rhd F^*)\,Z \cong \mu Z'.\, Z + D\,(Z'^2)$.

Take $FX = 1 + X^2$, then $F^*X \cong \mu X'.\, X + 1 + X'^2$. We can calculate $(F \star UD)\,Z \cong D\,1 + D\,(Z^2)$, therefore $(D \rhd F^*)\,Z \cong \mu Z'.\, Z + D\,1 + D\,(Z'^2)$.

These examples generalize to any wellpointed, locally presentable $\mathbb{C}$ with exponentials, when R and D are strong.

In exactly the same way as above, comonoid maps $D_0 \star D_1 \to G^\dagger$ are in bijection with maps $UD_0 \star UD_1 \to G$, and $\mathcal{C}(D_1, G^\dagger) \cong (UD_1 \mathrel{-\!\!\star} G)^\dagger$.

In the rest of this paper, we ignore comonad-comonad interaction laws and the internal hom of comonads since they are not our main focus. But developments similar to those for monad-comonad interaction laws and the Sweedler hom of monads and the Sweedler copower of a monad in Sects. 4, 5) below can be carried out for them as well.

4 Monad-comonad Interaction Laws (Co)algebraically

We now return to monad-comonad interaction laws specifically and explain the (co)algebraic perspective developed in [33]. (Props. 4 and 6 did not appear in [33].) First, monad-comonad interaction laws admit the following useful characterization in terms of (co)algebras of the (co)monads involved.

Proposition 3. *R-residual monad-comonad interaction laws ψ of T, D are in bijection with functors $\Psi : (\mathbf{Coalg}(D))^{\mathrm{op}} \times \mathbf{Alg}(R) \to \mathbf{Alg}(T)$ that internal-hom carriers, i.e., satisfy*

$$\begin{tikzcd} (\mathbf{Coalg}(D))^{\mathrm{op}} \times \mathbf{Alg}(R) \arrow[r, "\Psi"] \arrow[d, "U^{\mathrm{op}} \times U"'] & \mathbf{Alg}(T) \arrow[d, "U"] \\ \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \arrow[r, "\multimap"] & \mathbb{C} \end{tikzcd}$$

Proof (sketch). Given an interaction law ψ, the functor Ψ is defined by

$$\Psi((Y,\chi),(Z,\zeta)) = (Y \multimap Z,\ T(Y \multimap Z) \xrightarrow{\psi} DY \multimap RZ \xrightarrow{\chi \multimap \zeta} Y \multimap Z\ )$$

Conversely, given a functor Ψ, the corresponding interaction law ψ is defined by

$$\psi = \ T(Y \multimap Z) \xrightarrow{T(\varepsilon_Y \multimap \eta^R_Z)} T(DY \multimap RZ) \xrightarrow{\quad \xi \quad} DY \multimap RZ$$

where $(DY \multimap RZ, \xi) = \Psi((DY, \delta_Y), (RZ, \mu^R_Z))$. $\square$

We remark that such functors Ψ are completely determined by their action on (co)free (co)algebras. To be precise, there is a bijection between these functors and functors $\Psi' : (\mathbf{CoKl}(D))^{\mathrm{op}} \times \mathbf{Kl}(R) \to \mathbf{Alg}(T)$ that satisfy

$$\begin{tikzcd} (\mathbf{CoKl}(D))^{\mathrm{op}} \times \mathbf{Kl}(R) \arrow[r, "\Psi'"] \arrow[d, "K^{\mathrm{op}} \times K"'] & \mathbf{Alg}(T) \arrow[d, "U"] \\ \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \arrow[r, "\multimap"] & \mathbb{C} \end{tikzcd}$$

where $K : \mathbf{CoKl}(D) \to \mathbb{C}$ is the left adjoint of the coKleisli adjunction of D and $K : \mathbf{Kl}(R) \to \mathbb{C}$ is the right adjoint of the Kleisli adjunction of R.

The following reformulations of Prop. 1 enable a smooth derivation of further characterizations of monad-comonad interaction laws in terms of what we call runners, introduced next.

Corollary 1. *R-residual interaction laws of T, D are in bijection with functors $\Psi : \mathbf{Coalg}(D) \to [\mathbf{Alg}(R), \mathbf{Alg}(T)]^{\mathrm{op}}$ satisfying*

$$\begin{tikzcd} \mathbf{Coalg}(D) \arrow[rr, "\Psi"] \arrow[d, "U"'] & & {[\mathbf{Alg}(R), \mathbf{Alg}(T)]^{\mathrm{op}}} \arrow[d, "{[\mathbf{Alg}(R), U]^{\mathrm{op}}}"] \\ \mathbb{C} \arrow[r, "(Y \mapsto Y \multimap -)^{\mathrm{op}}"] & {[\mathbb{C}, \mathbb{C}]^{\mathrm{op}}} \arrow[r, "{[U, \mathbb{C}]^{\mathrm{op}}}"] & {[\mathbf{Alg}(R), \mathbb{C}]^{\mathrm{op}}} \end{tikzcd}$$

and also with functors $\Psi : \mathbf{Alg}(R) \to [\mathbf{Coalg}(D)^{\mathrm{op}}, \mathbf{Alg}(T)]$ *satisfying*

$$\begin{tikzcd}
\mathbf{Alg}(R) \arrow[rr, "\Psi"] \arrow[d, "U"'] & & {[(\mathbf{Coalg}(D))^{\mathrm{op}}, \mathbf{Alg}(T)]} \arrow[d, "{[(\mathbf{Coalg}(D))^{\mathrm{op}}, U]}"] \\
\mathbb{C} \arrow[r, "(Z \mapsto - \multimap Z)"] & {[\mathbb{C}^{\mathrm{op}}, \mathbb{C}]} \arrow[r, "{[U^{\mathrm{op}}, \mathbb{C}]}"] & {[(\mathbf{Coalg}(D))^{\mathrm{op}}, \mathbb{C}]}
\end{tikzcd}$$

Stateful Runners

Say that an R-residual *stateful runner* of T is an object $Y \in \mathbb{C}$ together with a family of maps

$$\theta_X : TX \otimes Y \to R(X \otimes Y)$$

natural in X satisfying

$$\begin{tikzcd}
X \otimes Y \arrow[r, equal] \arrow[d, "\eta_X \otimes Y"'] & X \otimes Y \arrow[d, "\eta^R_{X \otimes Y}"] \\
TX \otimes Y \arrow[r, "\theta_X"] & R(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
TTX \otimes Y \arrow[r, "\theta_{TX}"] \arrow[d, "\mu_X \otimes Y"'] & R(TX \otimes Y) \arrow[r, "R\theta_X"] & RR(X \otimes Y) \arrow[d, "\mu^R_{X \otimes Y}"] \\
TX \otimes Y \arrow[rr, "\theta_X"] & & R(X \otimes Y)
\end{tikzcd}$$

Maps $(Y, \theta) \to (Y', \theta')$ between stateful runners are maps $f : Y \to Y'$ satisfying $R(X \otimes f) \circ \theta_X = \theta'_X \circ (TX \otimes f)$. Stateful runners form a category $\mathbf{SRun}_R(T)$.

R-residual stateful runners of T with carrier Y are in bijection with monad maps $T \to \mathsf{St}^R_Y$ where St^R_Y is the R-transformed *state monad* for state object Y defined by $\mathsf{St}^R_Y X = Y \multimap R(X \otimes Y)$.

They are also in bijection with functors $\Theta : \mathbf{Alg}(R) \to \mathbf{Alg}(T)$ that internal-hom Y with the carrier, i.e., satisfy

$$\begin{tikzcd}
\mathbf{Alg}(R) \arrow[r, "\Theta"] \arrow[d, "U"'] & \mathbf{Alg}(T) \arrow[d, "U"] \\
\mathbb{C} \arrow[r, "Y \multimap -"] & \mathbb{C}
\end{tikzcd}$$

Proof (sketch). Given a stateful runner θ, the functor Θ is defined by

$$\Theta(Z, \zeta) = \; T(Y \multimap Z) \xrightarrow{\theta_{Y \multimap Z}} Y \multimap R((Y \multimap Z) \otimes Y) \xrightarrow{Y \multimap R\mathsf{ev}} Y \multimap RZ \xrightarrow{Y \multimap \zeta} Y \multimap Z$$

Conversely, given a functor Θ, the stateful runner θ is

$$\theta_X = \; TX \xrightarrow{T\mathsf{coev}} T(Y \multimap X \otimes Y) \xrightarrow{T(Y \multimap \eta^R_{X \otimes Y})} T(Y \multimap R(X \otimes Y)) \xrightarrow{\xi} Y \multimap R(X \otimes Y)$$

where $(Y \multimap R(X \otimes Y), \xi) = \Theta(R(X \otimes Y), \mu^R_{X \otimes Y})$. $\square$

This observation is strengthened by the following proposition that also talks about stateful runner maps.

Proposition 4. *The following is pullback square:*

$$\begin{tikzcd}
\mathbf{SRun}_R(T) \arrow[rr] \arrow[d, "U"'] & & {[\mathbf{Alg}(R), \mathbf{Alg}(T)]^{\mathrm{op}}} \arrow[d, "{[\mathbf{Alg}(R), U]^{\mathrm{op}}}"] \\
\mathbb{C} \arrow[r, "(Y \mapsto Y \multimap -)^{\mathrm{op}}"] & {[\mathbb{C}, \mathbb{C}]^{\mathrm{op}}} \arrow[r, "{[U, \mathbb{C}]^{\mathrm{op}}}"] & {[\mathbf{Alg}(R), \mathbb{C}]^{\mathrm{op}}}
\end{tikzcd}$$

Combining Prop. 4 with Cor. 1, we obtain a characterization of monad-comonad interaction laws in terms of stateful runners.

Proposition 5. *R-residual monad-comonad interaction laws T, D are in a bijection with functors Ψ : $\mathbf{Coalg}(D) \to \mathbf{SRun}_R(T)$ preserving carriers, i.e., satisfying*

$$\begin{tikzcd} \mathbf{Coalg}(D) \arrow[rr, "\Psi"] \arrow[dr, "U"'] & & \mathbf{SRun}_R(T) \arrow[dl, "U"] \\ & \mathbb{C} & \end{tikzcd}$$

Continuation-Based Runners

A D-fuelled *continuation-based runner* of T is an object $Z \in \mathbb{C}$ together with a family of maps

$$\theta_X : D(X \multimap Z) \to TX \multimap Z$$

natural in X satisfying

$$\begin{tikzcd} D(X \multimap Z) \arrow[r, "\theta_X"] \arrow[d, "\varepsilon_{X \multimap Z}"'] & TX \multimap Z \arrow[d, "\eta_{X \multimap Z}"] \\ X \multimap Z \arrow[r, equal] & X \multimap Z \end{tikzcd} \qquad \begin{tikzcd} D(X \multimap Z) \arrow[rr, "\theta_X"] \arrow[d, "\delta_{X \multimap Z}"'] & & TX \multimap Z \arrow[d, "\mu_{X \multimap Z}"] \\ DD(X \multimap Z) \arrow[r, "D\theta_X"] & D(TX \multimap Z) \arrow[r, "\theta_{TX}"] & TTX \multimap Z \end{tikzcd}$$

These runners form a category $\mathbf{CRun}_D(T)$.

D-fuelled continuation-based runners of T with carrier Z are in bijection with monad maps $T \to \mathsf{Cnt}^D_Z$, where Cnt^D_Z is the D-transformed *continuation monad* for answer object Z defined by $\mathsf{Cnt}^D_Z X = D(X \multimap Z) \multimap Z$.

Continuation-based runners are also in bijection with functors Θ : $(\mathbf{Coalg}(D))^{\mathrm{op}} \to \mathbf{Alg}(T)$ that internal-hom the carrier with Z, i.e., that satisfy

$$\begin{tikzcd} (\mathbf{Coalg}(D))^{\mathrm{op}} \arrow[r, "\Theta"] \arrow[d, "U^{\mathrm{op}}"'] & \mathbf{Alg}(T) \arrow[d, "U"] \\ \mathbb{C}^{\mathrm{op}} \arrow[r, "- \multimap Z"] & \mathbb{C} \end{tikzcd}$$

Moreover:

Proposition 6. *The following is a pullback square:*

$$\begin{tikzcd} \mathbf{CRun}_D(T) \arrow[rr] \arrow[d, "U"'] & & {[(\mathbf{Coalg}(D))^{\mathrm{op}}, \mathbf{Alg}(T)]} \arrow[d, "{[(\mathbf{Coalg}(D))^{\mathrm{op}}, U]}"] \\ \mathbb{C} \arrow[r, "Z \mapsto - \multimap Z"] & {[\mathbb{C}^{\mathrm{op}}, \mathbb{C}]} \arrow[r, "{[U^{\mathrm{op}}, \mathbb{C}]}"] & {[(\mathbf{Coalg}(D))^{\mathrm{op}}, \mathbb{C}]} \end{tikzcd}$$

Combining this proposition with Cor. 1, we obtain:

Proposition 7. *R-residual monad-comonad interaction laws of T, D are in bijection with functors Ψ : $\mathbf{Alg}(R) \to \mathbf{CRun}_D(T)$ that preserve carriers, i.e., that satisfy*

$$\begin{tikzcd} \mathbf{Alg}(R) \arrow[rr, "\Psi"] \arrow[dr, "U"'] & & \mathbf{CRun}_D(T) \arrow[dl, "U"] \\ & \mathbb{C} & \end{tikzcd}$$

5 Combining Sweedler Theory and the (Co)algebraic Perspective

We now combine our (co)algebraic observations with Sweedler theory.

Sweedler Hom

By definition, the Sweedler hom between monads T, R, if it exists, is the comonad $\mathcal{M}(T, R)$ together with an monad-comonad interaction law υ such that, for any other comonad D and monad-comonad interaction law ψ, there exists a unique comonad map $g : D \to \mathcal{M}(T, R)$ satisfying

$$\begin{tikzcd} TX \otimes DY \arrow[r, dotted, "TX \otimes g_Y"'] \arrow[rr, bend left, "\psi_{X,Y}"] & TX \otimes \mathcal{M}(T,R)Y \arrow[r, "\upsilon_{X,Y}"'] & R(X \otimes Y) \end{tikzcd}$$

Comonad maps $D \to D'$ are in bijection with functors $\mathbf{Coalg}(D) \to \mathbf{Coalg}(D')$ that preserve carriers. Therefore, by Prop. 5, the Sweedler hom, if it exists, is the comonad $\mathcal{M}(T, R)$ together with a carrier-preserving functor $\Upsilon : \mathbf{Coalg}(\mathcal{M}(T, R)) \to \mathbf{SRun}_R(T)$ such that, for any other comonad D and carrier-preserving functor $\Psi : \mathbf{Coalg}(D) \to \mathbf{SRun}_R(T)$, there exists a unique carrier-preserving functor $\Gamma : \mathbf{Coalg}(D) \to \mathbf{Coalg}(\mathcal{M}(T, R))$ such that

$$\begin{tikzcd} \mathbf{Coalg}(D) \arrow[r, dotted, "\Gamma"'] \arrow[rr, bend left, "\Psi"] \arrow[dr, bend right, "U"'] & \mathbf{Coalg}(\mathcal{M}(T,R)) \arrow[r, "\Upsilon"'] \arrow[d, "U"] & \mathbf{SRun}_R(T) \arrow[dl, "U"] \\ & \mathbb{C} & \end{tikzcd}$$

It follows that, if $(\mathbf{SRun}_R(T), U)$ is strictly comonadic, then $\mathcal{M}(T, R)$ exists and $(\mathbf{Coalg}(\mathcal{M}(T, R)), U) \cong (\mathbf{SRun}_R(T), U)$. (Should $(\mathbf{SRun}_R(T), U)$ fail to be strictly comonadic, then $\mathcal{M}(T, R)$ may still exist, but with different algebras.) Easy calculations show that U strictly creates equalizers of U-split pairs. Hence, by the dual of Beck's monadicity theorem, U is strictly comonadic if it is a left adjoint. Under our assumptions on $\mathbb{C}$, T and R from Sect. 2, all is well.

Theorem 1. *If $\mathbb{C}$ is locally presentable and T and R are accessible monads on $\mathbb{C}$, then $\mathbf{SRun}_R(T)$ is locally presentable and the forgetful functor $U : \mathbf{SRun}_R(T) \to \mathbb{C}$ is a left adjoint. Hence the Sweedler hom $\mathcal{M}(T, R)$ exists, is accessible, and satisfies $(\mathbf{Coalg}(\mathcal{M}(T, R)), U) \cong (\mathbf{SRun}_R(T), U)$.*

Proof (sketch). We first show that $\mathbf{SRun}_R(T)$ is locally presentable. The functor $U : \mathbf{SRun}_R(T) \to \mathbb{C}$ strictly creates colimits by easy calculations, and hence $\mathbf{SRun}_R(T)$ is cocomplete. For local presentability, it therefore remains to show that $\mathbf{SRun}_R(T)$ is accessible, which we do by appealing to the fact that accessible categories are closed under *inserters* and *equifiers*. The category of F-coalgebras, for any accessible endofunctor F on $\mathbb{C}$, is an inserter of accessible functors, and is therefore accessible by [1, Thm. 2.72]. For each Y, families of maps $\theta_X : TX \otimes Y \to R(X \otimes Y)$ natural in X are in bijection with maps $\chi : Y \to (T \mathbin{-\!\!\star} R)Y$, so that R-residual stateful runners of T are equivalently coalgebras (Y, χ) of the functor $T \mathbin{-\!\!\star} R$, satisfying two equations. One equation is an equality between two maps $Y \to (\mathsf{Id} \mathbin{-\!\!\star} R)Y$, the other between two maps $Y \to ((T \cdot T) \mathbin{-\!\!\star} R)Y$. It follows that $\mathbf{SRun}_R(T)$ is isomorphic to a full subcategory of the category $\mathbf{coalg}(T \mathbin{-\!\!\star} R)$ of $(T \mathbin{-\!\!\star} R)$-coalgebras, and that this full subcategory is the joint equifier of two natural transformations of accessible functors $\mathbf{coalg}(T \mathbin{-\!\!\star} R) \to \mathbf{coalg}(\mathsf{Id} \mathbin{-\!\!\star} R)$ and of two natural transformations of accessible functors $\mathbf{coalg}(T \mathbin{-\!\!\star} R) \to \mathbf{coalg}((T \cdot T) \mathbin{-\!\!\star} R)$. Accessible categories are closed under equifiers of natural transformations of accessible functors [1, Lemma 2.76], so $\mathbf{SRun}_R(T)$ is accessible and hence locally presentable.

As a colimit-preserving functor between locally presentable categories, U is a left adjoint by Freyd's special adjoint functor theorem, thus strictly comonadic. The induced comonad is the Sweedler hom $\mathcal{M}(T, R)$. Accessibility of $\mathcal{M}(T, R)$ follows from accessibility of the adjoints (the right adjoint by [1, Prop. 2.23]). $\square$

Example 3. Let $\mathbb{C} = \mathbf{Set}$. Take $TX = X^S$ (the reader monad for state object S). R-residual stateful runners of T are objects Y with families of maps $X^S \times Y \to R(X \times Y)$ natural in X or, equivalently, maps $Y \to R(S \times Y)$ constrained by two equations. For $R = \mathsf{Id}$ or $R = 1 + -$, these are in bijection with maps $Y \to S$. The comonad with such structured objects Y as coalgebras, which is the Sweedler hom of T and R, is $DY = S \times Y$ (the coreader monad for S). For a general accessible monad R, the Sweedler hom can be described as a subcomonad of the cofree comonad $DY = \nu Y'.\ Y \times R(S \times Y')$.

Take $TX = X^+ = \mu X'.X \times (1 + X')$ (the nonempty list monad with concatenation as multiplication, free semigroup monad). R-residual stateful runners of T are objects Y with families of maps $X^+ \times Y \to R(X \times Y)$ natural in X satisfying two equations or, equivalently, maps $(X \times X) \times Y \to R(X \times Y)$ constrained by one equation or, equivalently, maps $Y \to R(Y + Y)$ coassociative wrt. the coproduct monoidal structure of $\mathbf{Kl}(R)$, i.e., making Y into a cosemigroup. For $R = \mathsf{Id}$, the corresponding comonad is the cofree cosemigroup (wrt. the coproduct monoidal structure on $\mathbf{Set}$) comonad. Its underlying functor is $DY \cong Y \times (Y + Y)$.

These examples generalize to any wellpointed, locally presentable $\mathbb{C}$ with exponentials, when R is a strong monad.

Sweedler Copower

The Sweedler copower of a monad T by a comonad D, if it exists, is by definition the monad $D \rhd T$ together with a monad-comonad interaction law υ such that, for any other monad R and monad-comonad interaction law ψ, there exists a unique monad map $g : D \rhd T \to R$ satisfying

$$\begin{tikzcd} TX \otimes DY \arrow[r, "\upsilon_{X,Y}"'] \arrow[rr, bend left, "\psi_{X,Y}"] & (D \rhd T)(X \otimes Y) \arrow[r, dotted, "g_{X \otimes Y}"'] & R(X \otimes Y) \end{tikzcd}$$

Monad maps $R' \to R$ are in bijection with functors $\mathbf{Alg}(R) \to \mathbf{Alg}(R')$ that preserve carriers. Therefore, by Prop. 7, the Sweedler copower, if it exists, is the monad $D \rhd T$ together with a carrier-preserving functor $\Upsilon : \mathbf{Alg}(D \rhd T) \to \mathbf{CRun}_D(T)$ such that, for any other monad R and carrier-preserving functor $\Psi : \mathbf{Alg}(R) \to \mathbf{CRun}_D(T)$, there exists a unique carrier-preserving functor $\Gamma : \mathbf{Alg}(R) \to \mathbf{Alg}(D \rhd T)$ such that

$$\begin{tikzcd}
\mathbf{Alg}(R) \arrow[r, dotted, "\Gamma"'] \arrow[rr, bend left, "\Psi"] \arrow[dr, bend right, "U"'] & \mathbf{Alg}(D \rhd T) \arrow[r, "\Upsilon"'] \arrow[d, "U"] & \mathbf{CRun}_D(T) \arrow[dl, "U"] \\
& \mathbb{C} &
\end{tikzcd}$$

Consequently, if $(\mathbf{CRun}_D(T), U)$ is strictly monadic, then $D \rhd T$ exists and $(\mathbf{Alg}(D \rhd T), U) \cong (\mathbf{CRun}_D(T), U)$. This is the case as soon as U is a right adjoint by Beck's strict monadicity theorem, because U is easily verified to strictly create U-split coequalizers.

Theorem 2. *If $\mathbb{C}$ is locally presentable and T and D are accessible, then $\mathbf{CRun}_D(T)$ is locally presentable and the forgetful functor $U : \mathbf{CRun}_D(T) \to \mathbb{C}$ is a right adjoint. Hence the Sweedler copower $D \rhd T$ exists, is accessible, and satisfies $(\mathbf{Alg}(D \rhd T), U) \cong (\mathbf{CRun}_D(T), U)$.*

Proof (sketch). The proof is similar to that of Thm. 1. The functor U strictly creates limits, so $\mathbf{CRun}_D(T)$ is complete. The category $\mathbf{CRun}_D(T)$ is isomorphic to a full subcategory of the category of algebras of the functor $D \star T$, forming a joint equifier. Categories of algebras of accessible endofunctors on $\mathbb{C}$ are inserters of accessible functors, and hence form accessible categories. It follows that $\mathbf{CRun}_D(T)$ is also accessible, and hence locally presentable. The functor U strictly creates κ-filtered colimits, where κ is such that $\mathsf{Id} \star T$, $D \star T$, and $(D \cdot D) \star T$ are κ-accessible; in particular, U is accessible. Since U also strictly creates limits, it is therefore a right adjoint by [1, Theorem 1.66]. The induced monad is the Sweedler copower $D \rhd T$, which is accessible because both adjoints are. □

Example 4. Let $\mathbb{C} = \mathbf{Set}$. Take $TX = M \times X$ where $(M, u, *)$ is a monoid (the writer monad) and $DY = S \times Y$ (the coreader comonad). D-fuelled continuation-based runners of T are objects Z with families of maps $S \times Z^X \to Z^{M \times X}$ natural in X or, equivalently, maps $(S \times M) \times Z \to Z$, subject to two equations. The monad with such structured objects Z as algebras, which is the Sweedler copower of T and D, is the writer monad for the free monoid on $S \times M$ quotiented by $(s, a) * (s, b) = (s, a * b)$ and $u = (s, u)$.

6 Enriched Interaction Laws

In Sects. 2, 4, 5 above, we worked with (a full subcategory of) the category $[\mathbb{C}, \mathbb{C}]$ of endofunctors on a SMCC $\mathbb{C}$ and natural transformations between them, and abstracted it to a duoidal category $\mathbb{D}$ in Sect. 3.

An alternative is to proceed from an SMCC $(\mathbb{V}, \mathsf{I}, \otimes, \multimap)$ (copowered over itself by $\otimes$ and enriched and powered by $\multimap$) and another category $\mathbb{C}$ that is at least copowered or enriched over $\mathbb{V}$, or possibly both or even powered too. In this setting, a $\mathbb{V}$-*enriched functor-functor interaction law* is given by $\mathbb{V}$-enriched endofunctors F on $\mathbb{V}$ and G and H on $\mathbb{C}$ together with either a family of maps $\phi_{X,Y} : FX \bullet GY \to H(X \bullet Y)$ in $\mathbb{C}$ that are $\mathbb{V}$-natural in $X \in \mathbb{V}$ and $Y \in \mathbb{C}$ or, equivalently, a family of maps $\phi_{Y,Z} : F(\mathbb{C}(Y,Z)) \to \mathbb{C}(GY, HZ)$ in $\mathbb{V}$ that are $\mathbb{V}$-natural in $Y, Z \in \mathbb{C}$.

Two cases are of special interest.

- $\mathbb{V} = \mathbf{Set}$: Then the requirements that the category $\mathbb{C}$, the functors F, G, H and the natural transformation ϕ be $\mathbb{V}$-enriched are automatically met, but differently from the main setting of this paper, F is an endofunctor on a generally different category than G and H.
- $\mathbb{V} = \mathbb{C}$: Then the requirements that the functors F, G, H and the natural transformation ϕ be $\mathbb{V}$-enriched become real restrictions, but F, G, H remain endofunctors all on the same category.

The only case where the enriched setting agrees with the main one of this paper of Sects. 2–5, i.e., the concept of interaction law where there are no non-vacuous enrichment requirements and the endofunctors involved are all on the same category, is the intersection of the above two: $\mathbb{V} = \mathbb{C} = \mathbf{Set}$.

A more general situation in which the two settings do not differ too much is when $\mathbb{V} = \mathbb{C}$ and $\mathbb{C}$ is *monoidally wellpointed*. Then all functors with codomain $\mathbb{C}$ are uniquely $\mathbb{C}$-enriched (but may fail to admit an enrichment) and all natural transformations between $\mathbb{C}$-enriched functors with codomain $\mathbb{C}$ are $\mathbb{C}$-enriched.

In the case $\mathbb{V} = \mathbb{C}$, which is probably the most interesting case for mathematical semantics applications, the duoidal abstraction of Sect. 3 still applies. We can take $\mathbb{D}$ to be (a suitable full subcategory of) $\mathbb{C}$-$[\mathbb{C}, \mathbb{C}]$, where $\mathbb{C}$-$[\mathbb{C}, \mathbb{C}]$ is the ordinary category of $\mathbb{C}$-functors $\mathbb{C} \to \mathbb{C}$ (strong endofunctors).

In the case of a general $\mathbb{V}$, the simple duoidal abstraction ceases to apply. We need to switch to an action $\star : \mathbb{W} \times \mathbb{D} \to \mathbb{D}$ (in $\mathbf{MonCAT}_{\mathrm{oplax}}$) of a symmetric duoidal category $(\mathbb{W}, I_{\mathbb{W}}, \diamond_{\mathbb{W}}, J_{\mathbb{W}}, \star_{\mathbb{W}})$ on a monoidal category $(\mathbb{D}, I, \diamond)$ together with a functor $\multimap\!\!\star : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{W}$ such that $- \star G \vdash G \multimap\!\!\star -$ (in $\mathbf{CAT}$). Crucially, the action $\star$ comes with structural laws

$$I_{\mathbb{W}} \star I \to I \qquad (F \diamond_{\mathbb{W}} G) \star (H \diamond K) \to (F \star H) \diamond (G \star K)$$

witnessing oplaxity of $\star$. Similarly to the simple duoidal situation, we get that $\star$ and $\multimap\!\!\star$ lift to functors $\star : \mathbf{Comon}(\mathbb{W}) \times \mathbf{Comon}(\mathbb{D}) \to \mathbf{Comon}(\mathbb{D})$ and $\multimap\!\!\star : (\mathbf{Comon}(\mathbb{D}))^{\mathrm{op}} \times \mathbf{Mon}(\mathbb{D}) \to \mathbf{Mon}(\mathbb{W})$ and can then define measuring maps and Sweedler-like operations and ask if they are everywhere defined.

The instantiation is given by (suitable full subcategories of) $\mathbb{W} = \mathbb{V}$-$[\mathbb{V}, \mathbb{V}]$, $\mathbb{D} = \mathbb{V}$-$[\mathbb{C}, \mathbb{C}]$ and

$$\begin{aligned} (F \star G)\, Z &= \textstyle\int^{X,Y} \mathbb{C}(X \bullet Y, Z) \bullet (FX \bullet GY) &&= \textstyle\int^{Y} F(\mathbb{C}(Y,Z)) \bullet GY \\ (G \multimap\!\!\star H)\, X &= \textstyle\int_{Y,Z} (X \multimap \mathbb{C}(Y,Z)) \multimap \mathbb{C}(GY, HZ) &&= \textstyle\int_{Y} \mathbb{C}(GY, H(X \bullet Y)) \end{aligned}$$

where the integral signs now stand for $\mathbb{V}$-enriched coends and ends.

Runners as Generalized Algebras

Enriched monad-comonad interaction laws can be characterized as enriched functors between categories of (co)algebras analogously to Props. 3, 5, 7. But one pleasant feature of the enriched setting is that enriched versions of both stateful and continuation-based runners of T can be described as algebras of T in a generalized sense.

Suppose we are given an SMCC $\mathbb{V}$ (copowered over itself by $\otimes$ and enriched and powered by $\multimap$) and a $\mathbb{V}$-enriched monad T on $\mathbb{V}$. For a category $\mathbb{K}$ that is enriched and powered over $\mathbb{V}$, we say that an *algebra* of T *in* $\mathbb{K}$ as an object Y of $\mathbb{K}$ together with family of maps $\chi_X : X \pitchfork Y \to TX \pitchfork Y$ in $\mathbb{K}$ that is $\mathbb{V}$-enriched natural in $X \in \mathbb{V}$ and satisfies the equations

$$\begin{tikzcd} X \pitchfork Y \arrow[r, "\chi_X"] \arrow[dr, equal] & TX \pitchfork Y \arrow[d, "\eta_X \pitchfork Y"] \\ & X \pitchfork Y \end{tikzcd} \qquad \begin{tikzcd} X \pitchfork Y \arrow[r, "\chi_X"] \arrow[d, "\chi_X"'] & TX \pitchfork Y \arrow[d, "\mu_X \pitchfork Y"] \\ TX \pitchfork Y \arrow[r, "\chi_{TX}"] & TTX \pitchfork Y \end{tikzcd}$$

If $\mathbb{V}$ has enough limits, then these form a $\mathbb{V}$-category $\mathbf{Alg}(T, \mathbb{K})$, and there is a forgetful $\mathbb{V}$-functor $U : \mathbf{Alg}(T, \mathbb{K}) \to \mathbb{C}$. (The limits are required to carve out the object of algebra maps $(Y, \chi) \to (Y', \chi')$ from the hom-object $\mathbb{K}(Y, Y')$.)

An algebra like this is equivalently an object $Y \in \mathbb{K}$ together with a $\mathbb{V}$-enriched monad map $T \to \mathsf{Knt}_Y$ where $\mathsf{Knt}_Y X = \mathbb{K}(X \pitchfork Y, Y)$. If $\mathbb{V} = \mathbb{K}$, an algebra of T in this sense is the same as an algebra in the standard sense. In this case, we have $\mathsf{Knt}_Y X = (X \multimap Y) \multimap Y$.

Enriched runners of T turn out to be algebras of T in this generalized sense. Given a category $\mathbb{C}$ enriched and copowered over $\mathbb{V}$ and a $\mathbb{V}$-enriched monad R on $\mathbb{C}$, an $\mathbb{V}$-*enriched* R-residual *stateful runner* of T is an object $Y \in \mathbb{C}$ together with a family of maps $\theta_X : TX \bullet Y \to R(X \bullet Y)$ in $\mathbb{C}$ $\mathbb{V}$-natural in $X \in \mathbb{V}$ and satisfying two equations. Enriched stateful runners of T are in bijection with algebras of T in $(\mathbf{Kl}(R))^{\mathrm{op}}$.

Proof (sketch). The statement is wellformed since, as soon as $\mathbb{C}$ is $\mathbb{V}$-enriched and copowered by a functor $\bullet : \mathbb{V} \otimes \mathbb{C} \to \mathbb{C}$, we have that $\mathbf{Kl}(R)$ is $\mathbb{V}$-enriched and copowered by a functor $\mathbb{V} \otimes \mathbf{Kl}(R) \to \mathbf{Kl}(R)$ that agrees with $\bullet$ on objects. Therefore $(\mathbf{Kl}(R))^{\mathrm{op}}$ is $\mathbb{V}$-enriched and powered by the opposite of that functor. We have the following chain of bijections:

$$\begin{array}{c} TX \bullet Y \to R(X \bullet Y) \text{ in } \mathbb{C} \text{ } \mathbb{V}\text{-nat. in } X \\ \hline\hline TX \bullet Y \to X \bullet Y \text{ in } \mathbf{Kl}(R) \text{ } \mathbb{V}\text{-nat. in } X \\ \hline\hline X \pitchfork Y \to TX \pitchfork Y \text{ in } (\mathbf{Kl}(R))^{\mathrm{op}} \text{ } \mathbb{V}\text{-nat. in } X \end{array}$$ □

The statement about the category of enriched stateful runners is:

Proposition 8. *If* $\mathbf{Alg}(T, (\mathbf{Kl}(R))^{\mathrm{op}})$ *exists as a* $\mathbb{V}$*-category, then so does* $\mathbb{V}\text{-}\mathbf{SRun}_R(T)$*, and the following is a pullback square (in* $\mathbb{V}\text{-}\mathbf{CAT}$*).*

$$\begin{tikzcd} \mathbb{V}\text{-}\mathbf{SRun}_R(T) \arrow[r] \arrow[d, "U"'] & (\mathbf{Alg}(T, (\mathbf{Kl}(R))^{\mathrm{op}})^{\mathrm{op}} \arrow[d, "U^{\mathrm{op}}"] \\ \mathbb{C} \arrow[r, "J"] & \mathbf{Kl}(R) \end{tikzcd}$$

In the special case when $\mathbb{V} = \mathbb{C}$ and $R = \mathsf{Id}_{\mathbb{C}}$, we get $(\mathbf{Coalg}(\mathcal{M}^{\mathbb{C}}(T, \mathsf{Id}), U) \cong ((\mathbf{Alg}(T, \mathbb{C}^{\mathrm{op}}))^{\mathrm{op}}, U^{\mathrm{op}})$ ("coalgebras" of the $\mathbb{C}$-monad T).

By the same token, given a $\mathbb{V}$-enriched and powered category $\mathbb{C}$ and a $\mathbb{V}$-enriched comonad D on $\mathbb{C}$, we can define what an $\mathbb{V}$-*enriched* D-fuelled *continuation based runner* of T is: an object $Z \in \mathbb{C}$ together with a family of maps $\theta_X : D(X \pitchfork Z) \to TX \pitchfork Z$ in $\mathbb{C}$ that is $\mathbb{V}$-natural in $X \in \mathbb{V}$ and satisfies two equations. Enriched continuation-based runners of T are in bijection with algebras of T in the coKleisli category of D. Moreover:

Proposition 9. *If* $\mathbf{Alg}(T, \mathbf{CoKl}(D))$ *exists as a* $\mathbb{V}$*-category, then so does* $\mathbb{V}$-$\mathbf{CRun}_D(T)$*, and the following is a pullback square:*

$$\begin{tikzcd}
\mathbb{V}\text{-}\mathbf{CRun}_D(T) \arrow[r] \arrow[d, "U"'] & \mathbf{Alg}(T, \mathbf{CoKl}(D)) \arrow[d, "U"] \\
\mathbb{C} \arrow[r, "J"] & \mathbf{CoKl}(D)
\end{tikzcd}$$

7 Related Work

In semantics work, the use of monads as notions of computation was pioneered by Moggi [23], but the first to study comonads (or algebraic theories comodelled) as notions of environment (not under that name) were Shkaravska and Power [29]. This work was developed further by Plotkin and Power [24] and then Møgelberg and Staton [22] (who considered the enriched setting). Stateful runners appeared in Uustalu's paper [32], who noticed that nonresidual stateful runners of a set monad induced by an algebraic theory are in bijection with coalgebras of the comonad induced by the same theory (comodels). The concept of monad-comonad interaction law was distilled by Katsumata et al. [16], who also noticed that the universal interacting comonad of a monad is an instance of the Sweedler hom from Sweedler theory for duoidal categories; they calculated the dual and Sweedler dual for a number of cases. Uustalu and Voorneveld [33] noticed the bijection between monad-comonad interaction laws and suitable functors between categories of (co)algebras and that, in addition to stateful runners, monad-comonad interaction laws relate to continuation-based runners. Garner [12,11] further developed this thread. In particular, he gave a formula for the Sweedler duals of polynomial monads, and demonstrated properties of the dual/Sweedler dual (costructure/cosemantics) adjunction for accessible **Set**-(co)monads, such as its idempotency. He also pointed out that, when T and R are accessible **Set**-monads, the coalgebras of the Sweedler hom $\mathcal{M}(T, R)$ are algebras of T in $(\mathbf{Kl}(R))^{\mathrm{op}}$ with, as maps between them, maps in **Set** that $J : \mathbf{Set} \to \mathbf{Kl}(R)$ sends to algebra maps.

Independently, and earlier than in the semantics community, monad-comonad interaction laws were discovered among functional programmers by Kmett [19] and Freeman [8].

There is a disconnected and more mature thread of work in universal algebra started by Freyd [9] (or even Kan [15]), and continued by Tall and Wraith

[31,34] and Bergman and Hausknecht [5], studying functors from coalgebras of a covariety to algebras (like those of our Prop. 3) in the case $\mathbb{V} = \mathbf{Set}$, $R = \mathsf{Id}_{\mathbb{C}}$ of our enriched setting. (There are also textbook expositions, by Popescu and Popescu [26, Ch. 3] and Bergman [4, Ch. 10].) Strangely, this thread seems to have never been picked up in semantics work. It was not cited in the work by Power and coauthors [29,24], and the later authors (except Garner) have been unaware of it.

Sweedler's original work [30] was for (co)algebras over a field. Anel and Joyal [3] studied the Sweedler theory in great detail for dg-(co)algebras [3]. It was abstracted for (co)monoids in symmetric monoidal closed categories by Porst and Street [28] and Hyland et al. [14] (the internal hom of comonoids is older and goes back to Porst [27]) and then generalized for duoidal categories by López Franco and Vasilakopoulou [20]. A typical example duoidal structure on a functor category is given by the Day convolution and pointwise tensor. Garner and López Franco [13] considered the example of composition and the Day convolution of endofunctors (κ-accessible for a fixed κ).

We do not know the earliest reference to generalized algebras of a monad, in particular, coalgebras of a monad. The latter were considered by Poinsot and Porst [25] (and models of algebraic theories elsewhere than **Set** are standard).

8 Conclusion and Future Work

We have studied universal (co)monads for monad-comonad interactions. We have shown that an elegant setting for such a study on a more general level is provided by Sweedler theory for general duoidal categories as developed by López Franco and Vasilakopoulou [20]. But for results about monad-comonad interaction specifically it is fruitful to combine it with the (co)algebraic perspective on monad-comonad interaction laws [33]. This makes it possible to characterize the universal (co)monads defined by Sweedler operations via their categories of (co)algebras in terms of different flavors of runners.

We have witnessed that there is the choice of whether to work with ordinary monad-comonad interaction laws or with the enriched version. It remains to be seen which option yields a richer or more useful theory. An issue with the enriched option is that we know little about accessibility for enriched categories, although some studies exist (e.g., [18,6,7]).

We refrained from discussing it in this paper altogether, but of course one can specifically study interaction laws of monads and comonads specified by algebraic theories. We intend to do this in a sequel paper. We also plan to explain properly the significance for semantics of the constructions of this paper by describing in detail how they work on semantics-motivated examples and what this means.

Acknowledgements We thank Niels Voorneveld for many useful discussions. Richard Garner's work is an endless source of inspiration. D.M. and T.U. were supported by the Icelandic Research Fund project grant no. 196323-053, T.U. also by the Estonian Research Council team grant no. PRG1210. E.R. was supported by the Estonian Research Council personal grant no. PSG659.

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