Flexibly Graded Monads and Graded Algebras

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Abstract. When modelling side-effects using a monad, we need to equip the monad with effectful operations. This can be done by noting that each algebra of the monad carries interpretations of the desired operations. We consider the analogous situation for graded monads, which are a generalization of monads that enable us to track quantitative information about side-effects. Grading makes a significant difference: while many graded monads of interest can be equipped with similar operations, the algebras often cannot. We explain where these operations come from for graded monads. To do this, we introduce the notion of flexibly graded monad, for which the situation is similar to the situation for ordinary monads. We then show that each flexibly graded monad induces a canonical graded monad in such a way that operations for the flexibly graded monad carry over to the graded monad. In doing this, we reformulate grading in terms of locally graded categories, showing in particular that graded monads are a particular kind of relative monad. We propose that locally graded categories are a useful setting for work on grading in general.

Keywords: graded monad · graded algebra · flexible grading · relative monad · computational effect · locally graded category

1 Introduction

Computational effects are often modelled, following Moggi [18,19], using (strong) monads. The structure of the monad is used to interpret sequencing of computations, but to interpret the constructs that cause effects we need additional data—usually a collection of *algebraic operations* in the sense of Plotkin and Power [21]. For example, finite nondeterminism can be modelled using the usual list monad on **Set**; nullary and binary nondeterministic choice are modelled as the empty list and concatenation of lists. *Presentations* of theories corresponding to monads are an important source of these algebraic operations. For a given presentation, an *algebra* consists of an object together with interpretations of its operations, subject to its equations. The corresponding monad T (if it exists) is defined to be such that the T-algebras are the algebras of the presentation. Every T-algebra therefore admits interpretations of the operations of the presentation, and for free T-algebras these interpretations give rise to algebraic operations in the sense of Plotkin and Power. For example, if we start with the presentation of monoids with a constant and a binary operation and the unitality and associativity equations, then T will be the list monad; free T-algebras have lists as carriers, and the empty list and concatenation of lists provide the monoid structure of these free algebras.

We consider the analogous situation for the *graded* monads of Smirnov [22], Melliès [16] and Katsumata [9], focusing in particular on their application to tracking quantitative information about effects of programs [9,20]. (There are other applications, such as in process semantics [2,17], and in probability theory [3].) Instead of assigning a single object TX to each object X, a graded monad assigns an object TXe to each object X and *grade* e. The quantitative information is represented by e. For example, the grades could be natural numbers, upper-bounding the number of alternative outcomes from nondeterministic computations. We can model these computations using the graded list monad $\mathsf{T} = \mathsf{List}$ where $TXe = \mathrm{List}Xe$ is the set of lists over X of length at most e.

At first glance, the situation with operations for graded monads seems similar to the situation for ordinary monads. The empty list $() \in \mathrm{List}X0$ and concatenation $\mathrm{List}Xe_1 \times \mathrm{List}Xe_2 \to \mathrm{List}X(e_1 + e_2)$ make $\mathrm{List}X$ into a *graded monoid*. We might expect that this graded monoid structure arises because List-algebras are graded monoids. But this is not the case, as we show below: graded monoids are not the algebras of the graded monad List, or indeed of *any* graded monad. The same phenomenon occurs also with other examples: the free algebras TX of a graded monad can often be equipped with some algebraic structure of interest, even when the general algebras cannot. (We also consider *graded arithmoids* as an example below.)

We explain this phenomenon via a new notion of *flexibly graded monad*. Flexibly graded monads should be thought of as more general than graded monads (though constructing a flexibly graded monad with the same algebras as a given graded monad relies on existence of certain colimits). The point is that flexibly graded monads often do capture these algebraic structures, for example, there *is* a flexibly graded monad $\mathsf{List}_{\mathsf{flex}}$ whose algebras are graded monoids. We show that every flexibly graded monad T induces a graded monad $\lfloor \mathsf{T} \rfloor$; the latter may not have the same algebras as T, but does satisfy a universal property (Lemma 2) formulated in terms of algebras. Moreover, every free $\lfloor \mathsf{T} \rfloor$-algebra forms a T-algebra. For example, we obtain List as the graded monad $\lfloor \mathsf{List}_{\mathsf{flex}} \rfloor$, and hence also the graded monoid structure of the free algebras $\mathrm{List}X$. This paper should be viewed as a step towards developing notions of presentation and algebraic operation for graded monads that can include, for example, the operations of a graded monoid. The graded presentations considered from the literature [22,17,2,12] are not flexible enough to present graded monoids. The appropriate notions of flexibly graded presentation and flexibly graded algebraic operation are discussed in the sequel paper [10].

As part of the development, we formulate graded monads in terms of *locally graded categories* of Wood [26]. (He used a different name, we use Levy's [13] terminology). These are a particular instance of enriched categories, and so they enable us to use constructions and results that apply to enriched categories in general. Still, here we use an explicit description of locally graded categories to avoid assuming knowledge of enriched category theory. We show that graded monads and flexibly graded monads are just instances of *relative monads* [1] on functors between locally graded categories; we rely heavily on general facts about relative monads in our other results. Locally graded categories also enable us to simplify some previous work (such as Fujii et al.'s [4], which uses *actegories* instead). For this reason, we propose that locally graded categories are a useful setting for work on grading in general.

Contributions We begin by reviewing the existing notions of graded monad (Section 2) and locally graded category (Section 3). We then do the following.

- We define the appropriate notion of *relative monad* for locally graded categories, and develop some of the associated theory (Section 4). We show that graded monads are relative monads, and introduce our notion of *flexibly graded monad*. We show that flexibly graded monads capture algebraic structures we are interested in, such as graded monoids.
- We show that every flexibly graded monad T induces a graded monad $\lfloor \mathsf{T} \rfloor$ satisfying a universal property (Section 5). This construction canonically equips free $\lfloor \mathsf{T} \rfloor$-algebras with additional structure (for example, equips $\mathrm{List}\, X$ with the structure of a graded monoid).
- We discuss the reverse direction: that of constructing a canonical flexibly graded monad $\lceil \mathsf{T} \rceil$ from a graded monad T (Section 6). We use this to show that graded monads do not capture certain algebraic structures (e.g. graded monoids), we characterize the existence of $\lceil \mathsf{T} \rceil$ in terms of existence of certain colimits.

2 Graded Monads

We begin by reviewing the existing notion of graded monad. The grades e are the objects of a category $\mathbb{E}$, one example being the poset $\mathbb{N}_{\leq}$ of natural numbers with their usual ordering. Various other examples can be found in the literature on graded monads.

Definition 1. *An* $\mathbb{E}$-graded object *of* $\mathbb{C}$*, where* $\mathbb{E}$ *and* $\mathbb{C}$ *are categories, is a functor* $X : \mathbb{E} \to \mathbb{C}$*. These form a category* $[\mathbb{E}, \mathbb{C}]$*, with natural transformations as morphisms.*

To assign suitable grades to the unit and Kleisli extension of a graded monad, we need a unit grade 1 and multiplication operator $(\cdot)$ on grades. For the rest of the paper, we suppose a monoidal category $(\mathbb{E}, 1, \cdot)$ that we assume to be small (for technical reasons) and strict (for convenience). For example, multiplication of natural numbers makes $\mathbb{N}_{\leq}$ into a strict monoidal category $\mathbb{N}_{\leq}^{\times} = (\mathbb{N}_{\leq}, 1, \cdot)$. We often omit the prefix $\mathbb{E}$- from $\mathbb{E}$-graded.

Definition 2 ([22,16,9]). *An* $\mathbb{E}$-graded monad T *on a category* $\mathbb{C}$ *consists of the following data:*

– *a graded object* $TX : \mathbb{E} \to \mathbb{C}$ *for each* $X \in |\mathbb{C}|$*;*
– *a* unit *morphism* $\eta_X : X \to TX1$ *for each* $X \in |\mathbb{C}|$*;*
– *a* Kleisli extension *operator* $(-)^\dagger$ *that maps every morphism* $f : X \to TYe$ *and grade* $d \in |\mathbb{E}|$ *to a morphism* $f^\dagger_d : TXd \to TY(d \cdot e)$.

Kleisli extension is required to be natural in d *and* e*, and to satisfy the following unit and associativity laws.*

$$\begin{aligned} f^\dagger_1 \circ \eta_X &= f && \text{for each } f : X \to TYe \\ \mathrm{id}_{TXd} &= (\eta_X)^\dagger_d && \text{for each } X \in |\mathbb{C}|, d \in |\mathbb{E}| \\ (g^\dagger_e \circ f)^\dagger_d &= g^\dagger_{d \cdot e} \circ f^\dagger_d && \text{for each } f : X \to TYe, g : Y \to TZe', d \in |\mathbb{E}| \end{aligned}$$

2.1 Examples

We use the following three examples throughout the paper. In each case, we define a graded monad T, and then show that the T arises canonically from some class of (graded) algebraic structures. The latter fact provides a way of equipping the free algebras TX with the corresponding algebraic structure.

– The graded monad List (Definition 3) arises canonically from graded monoids (Definition 4), and so $\mathrm{List}X$ forms a graded monoid. Despite this, graded monoids are not the algebras for any graded monad (Theorem 4).
– For each graded monoid M, the *graded writer monad* Wr^{M} (Definition 5) arises canonically from M-*actions* (Definition 6). Differently from the List example, Wr^{M}-algebras are exactly M-actions (Example 7).
– We define a graded monad Count for modelling computations that increment and decrement a counter (Definition 7). In this case, the corresponding algebraic structure is our notion of *graded arithmoid* (Definition 8), so $\mathrm{Count}X$ forms a graded arithmoid for each set X. Graded arithmoids are not the algebras for Count or for any other graded monad (Theorem 5).

In this section, we define each of the graded monads, and the corresponding algebraic structure.

Graded Monoids and List We begin with the example of the graded list monad.

Definition 3. *The* $\mathbb{N}^\times_\leq$*-graded monad* List *on* **Set** *maps each set* X *to the graded object* $\mathrm{List}X$ *of lists over* X *of bounded length:* $\mathrm{List}Xe$ *is the set of lists of length at most* $e \in \mathbb{N}$*, and for* $e \leq e' \in \mathbb{N}$ *the function* $\mathrm{List}X(e{\leq}e') : \mathrm{List}Xe \to \mathrm{List}Xe'$ *is the inclusion (where we write* $e{\leq}e'$ *for the unique element of* $\mathbb{N}_\leq(e, e')$*). The unit* $\eta_X : X \to \mathrm{List}X1$ *and Kleisli extension* $f^\dagger_d : \mathrm{List}Xd \to \mathrm{List}Y(d \cdot e)$ *of* $f : X \to \mathrm{List}Ye$ *are similar to those of the usual list monad on* **Set***: they are defined by*

$$\eta_X x = (x) \qquad f^\dagger_d(x_1, \ldots, x_k) = fx_1 +\!\!+ \cdots +\!\!+ fx_k$$

where $(+\!\!+)$ *is concatenation of lists.*

For every set X, the graded object $\mathrm{List}X$ forms a *graded monoid* in the sense of the following definition, with

$$() \in \mathrm{List}X0 \qquad (+\!\!+) : \mathrm{List}Xe_1 \times \mathrm{List}Xe_2 \to \mathrm{List}X(e_1 + e_2)$$

as unit and multiplication.

Definition 4. *We write $\mathbb{N}^+_\leq$ for the strict monoidal category $\mathbb{N}^+_\leq = (\mathbb{N}_\leq, 0, +)$. A $\mathbb{N}^+_\leq$-*graded monoid *$\mathsf{A} = (A, u, m)$ on* **Set** *consists of a graded object $A : \mathbb{N}_\leq \to$* **Set** *(the* carrier*), a* unit *element $u \in A0$, and a family of* multiplication *functions $m_{e_1,e_2} : Ae_1 \times Ae_2 \to A(e_1 + e_2)$ natural in $e_1, e_2 \in \mathbb{N}_\leq$, such that multiplication is unital and associative:*

$$m_{0,e}(u, x) = x = m_{e,0}(x, u)$$
$$m_{e_1+e_2,e_3}(m_{e_1,e_2}(x, y), z) = m_{e_1,e_2+e_3}(x, m_{e_2,e_3}(y, z))$$

A homomorphism *$h : \mathsf{A} \to \mathsf{A}'$ is a natural transformation $h : A \Rightarrow A'$ such that*

$$h_0 u = u \qquad h_{e_1+e_2}(m_{e_1,e_2}(x, y)) = m_{e_1,e_2}(h_{e_1}x, h_{e_2}y)$$

(This definition can easily be generalized to grades other than natural numbers with addition and to monoidal categories other than **Set**, but for simplicity we consider only $\mathbb{N}^+_\leq$-graded monoids in **Set**.) An example is the following grading of the additive monoid of natural numbers:

$$Ne = \{0, \ldots, e\} \quad N(e{\leq}e')\, n = n \qquad u = 0 \qquad m_{e_1,e_2}(n_1, n_2) = n_1 + n_2$$

We give an informal explanation of why the algebras of the graded monad List are not graded monoids (for the proof, see Theorem 4). A List-*algebra* consists of a carrier $A : \mathbb{N}_\leq \to \mathbf{Set}$, and an operator $(-)^\ddagger$ that maps functions to functions as follows:

$$\frac{f : X \to Ae}{f^\ddagger_d : \mathsf{List}Xd \to A(d \cdot e)}$$

These are required to satisfy some laws; we defer the full definition to Section 4.1. Every graded monoid induces a List-algebra, by defining

$$f^\ddagger_d[x_1, \ldots, x_k] = m(fx_1, m(fx_2, \cdots m(fx_{k-1}, m(fx_k, u)) \cdots))$$

(where we omit the subscripts of m). If List-algebras were graded monoids, then this construction would be a bijection (by Corollary 1 below), so we should be able to recover m from $(-)^\ddagger$. This is not the case, intuitively because $f^\ddagger_d[x_1, \ldots, x_k]$ is an iterated multiplication of elements of A that all have the same grade, while the two arguments of m can have different grades. For a concrete example, we cannot recover the additive graded monoid structure on N above from

$$f^\ddagger_d[x_1, \ldots, x_k] = fx_1 + \cdots + fx_k$$

To see why, let N' be the smallest family of subsets $N'e \subseteq Ne$ closed under the inclusions $N(e'{\leq}e'')$ and under $(-)^\ddagger$, and such that $2 \in N'2$ and $3 \in N'3$. The List-algebra structure on N restricts to N', so we can recover m only if m also restricts to N'. But it does not, because m sends $(2, 3) \in N'2 \times N'3$ to $5 \notin N'5$.

M-actions and Wr^{M} Our second example is the following.

Definition 5. *Every $\mathbb{N}^{+}_{\leq}$-graded monoid $\mathsf{M} = (M, u, m)$ induces a $\mathbb{N}^{+}_{\leq}$-graded* writer *monad Wr^{M}, with assignment on objects, unit, and Kleisli extension defined by*

$$\mathrm{Wr}^{\mathsf{M}} X e = M e \times X \qquad \eta_X x = (u, x)$$
$$f^{\dagger}_{d}(p, x) = \mathrm{let}\ (q, y) = f x\ \mathrm{in}\ (m_{d,e}(p, q), y)$$

For every set X, the graded monoid M *acts* on $\mathrm{Wr}^{\mathsf{M}} X$ via the multiplication of M. Precisely, if we define

$$\mathrm{act}_{e_1,e_2} : M e_1 \times \mathrm{Wr}^{\mathsf{M}} X e_2 \to \mathrm{Wr}^{\mathsf{M}} X (e_1 + e_2)$$
$$\mathrm{act}_{e_1,e_2}(p, (q, y)) = (m_{e_1,e_2}(p, q), y)$$

then $(\mathrm{Wr}^{\mathsf{M}} X, \mathrm{act})$ is an M-action in the following sense.

Definition 6. *Let M be a $\mathbb{N}^{+}_{\leq}$-graded monoid. An* M-action *is a pair $\mathsf{A} = (A, \mathrm{act})$ of a graded object $A : \mathbb{N}_{\leq} \to \mathbf{Set}$ and a natural family of functions $\mathrm{act}_{e_1,e_2} : M e_1 \times A e_2 \to A(e_1 + e_2)$ satisfying*

$$\mathrm{act}_{0,e}(u, x) = x \quad \mathrm{act}_{e_1+e_2,e_3}(m_{e_1,e_2}(p, q), x) = \mathrm{act}_{e_1,e_2+e_3}(p, \mathrm{act}_{e_2,e_3}(q, x))$$

A homomorphism *$h : \mathsf{A} \to \mathsf{A}'$ of M-actions is a natural transformation $h : A \Rightarrow A'$ such that*

$$h_{e_1+e_2}(\mathrm{act}_{e_1,e_2}(p, x)) = \mathrm{act}_{e_1,e_2}(p, h_{e_2} x)$$

Graded Arithmoids and Count Our third example is computations that interact with a counter (which stores a natural number). These computations are able to either return a value, without changing the counter, or to do one of the following two operations.

- Increment: increase the value of the counter by 1, and then continue with a given computation.
- Test and decrement: if the value is 0, then continue with one computation, otherwise decrease the value by 1 and continue with another computation.

This can be seen as a special case of interaction with a stack of values drawn from a set V, in the case $V = 1$ (the stack is determined by its size, which is the value of the counter). Increment and decrement respectively correspond to push and pop. The graded monad is a graded version of Goncharov's stack monad [7], specialized to $V = 1$, and our notion of graded arithmoid (Definition 8 below) similarly arises by grading Goncharov's presentation of the stack monad.

We only consider finite computations, and in particular each computation can test the value of the counter only finitely many times (in other words, can interact with only a finite prefix of the stack, whose size depends on the computation). As a consequence, computations cannot always learn the exact value of the counter. This restriction is captured by the conditions involving ρ below.

Grades are integers, which provide an upper bound on the net amount the counter increases. (A negative upper bound $-e$ is equivalently a lower bound e on the amount the counter decreases by.) For example, if the counter is initially 6 and we run a computation of grade 3, then the final value will be at most 9 (but intermediate values can be greater than 9).

Definition 7. *We write $\mathbb{Z}_{\leq}$ for the poset of integers with their usual ordering, which forms a strict monoidal category $\mathbb{Z}^{+}_{\leq} = (\mathbb{Z}_{\leq}, 0, +)$ using addition of integers. The $\mathbb{Z}^{+}_{\leq}$-graded monad* Count *on* **Set** *is defined as follows. Given a set X, the graded object* $\mathrm{Count}X$ *is given by*

$$\begin{aligned}\mathrm{Count}X e = \{t : \textstyle\prod_{i:\mathbb{N}} [0..i+e] \times X \mid \\ \exists \rho \in \mathbb{N}.\, \forall k, j \in \mathbb{N}, x \in X.\, t\,\rho = (j, x) \Rightarrow t(\rho + k) = (j + k, x)\}\end{aligned}$$

where $[0..n] = \{0, 1, \ldots, n\}$ (empty for negative n). Thus computations t are dependent functions that map each initial counter value i to a pair (j, x) of a final counter value j such that $j - i \leq e$ and a result x. (There are no such dependent functions if $e < 0$, i.e. $\mathrm{Count}X e$ *is empty in this case.) The unit of the graded monad leaves the counter unchanged, and the Kleisli extension uses the final counter value of one computation as the initial counter value of another:*

$$\eta_X x = \lambda i.\,(i, x) \qquad f^{\dagger}_{d}\, t = \lambda i.\,\mathrm{let}\,(j, x) = t\,i \text{ in } f\,x\,j$$

The increment and decrement operations described above are captured by the following functions:

$$\begin{array}{ll}\mathrm{inc}_e : \mathrm{Count}X e \to \mathrm{Count}X(e{+}1) & \mathrm{dec}_e : \mathrm{Count}X e \times \mathrm{Count}X(e{+}1) \to \mathrm{Count}X e \\ \mathrm{inc}_e\, t = \lambda i.\, t(i+1) & \mathrm{dec}_e(t_1, t_2) = \lambda i.\,\text{if } i{=}0 \text{ then } t_1 0 \text{ else } t_2(i{-}1)\end{array}$$

and these form *graded arithmoids* in the following sense.

Definition 8. *A* graded arithmoid *is a triple $\mathsf{A} = (A, \mathrm{inc}, \mathrm{dec})$ of a graded object $A : \mathbb{Z}_{\leq} \to$* **Set** *and natural families of functions*

$$\mathrm{inc}_e : Ae \to A(e+1) \qquad \mathrm{dec}_e : Ae \times A(e+1) \to Ae$$

satisfying

$$\mathrm{inc}_e(\mathrm{dec}_e(x, y)) = y \qquad \mathrm{dec}_e(x, \mathrm{inc}_e x) = x$$
$$\mathrm{dec}_e(\mathrm{dec}_e(x, y), z) = \mathrm{dec}_e(x, z)$$

A homomorphism *$h : \mathsf{A} \to \mathsf{A}'$ of graded arithmoids is a natural transformation $h : A \Rightarrow A'$ such that*

$$h_{e+1}(\mathrm{inc}_e x) = \mathrm{inc}_e(h_e x) \quad h_e(\mathrm{dec}_e(x, y)) = \mathrm{dec}_e(h_e x, h_{e+1} y)$$

3 Locally Graded Categories

Locally graded categories are similar to ordinary categories, except that each morphism has a *grade* e in addition to a domain and codomain. An example of this situation appeared already in the definition of graded monad. While morphisms $f : X \Rightarrow Y$ in the ordinary category $[\mathbb{E}, \mathbb{C}]$ preserve the grades of elements (f sends elements of Xd to elements of Yd), Kleisli extensions $f^\dagger : TX \Rightarrow TY(-\cdot e)$ multiply by a grade e; in the locally graded category $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$ of graded objects (Definition 10), $f^\dagger$ is a morphism from TX to TY *of grade* e.

Definition 9 ([26]). *A* locally $\mathbb{E}$-graded category $\mathcal{C}$ *consists of*

- *a collection* $|\mathcal{C}|$ *of* objects*;*
- *for each* $X, Y \in |\mathcal{C}|$ *and* $e \in |\mathbb{E}|$*, a set* $\mathcal{C}(X, Y)e$ *of* morphisms *from* X *to* Y *of grade* e*; we write* $f : X -e\!\!\rightarrow Y$ *to indicate* $f \in \mathcal{C}(X, Y)e$*;*
- *for each* $X \in |\mathcal{C}|$*, a morphism* $\mathrm{id}_X : X -1\!\!\rightarrow X$*;*
- *for each* $f : X -e\!\!\rightarrow Y$ *and* $g : Y -e'\!\!\rightarrow Z$*, a morphism* $g \circ f : X -e\cdot e'\!\!\rightarrow Z$*;*
- *for each* $\zeta \in \mathbb{E}(e, e')$ *and* $f : X -e\!\!\rightarrow Y$*, a morphism* $\zeta^* f : X -e'\!\!\rightarrow Y$ *(the* coercion *of* f *along* ζ*);*

such that composition is unital ($\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$*) and associative (*$(h \circ g) \circ f = h \circ (g \circ f)$*); coercions are functorial (*$\mathrm{id}_e^* f = f$ *and* $\xi^*(\zeta^* f) = (\xi \circ \zeta)^* f$*); and such that composition commutes with coercion (*$(\xi \cdot \zeta)^*(g \circ f) = \xi^* g \circ \zeta^* f$*).*

(In Wood's terminology [26, Definition 1.1], these are *large* $\mathbb{E}^{\mathrm{op}}$*-categories*.) We systematically use blackboard bold letters like $\mathbb{C}$ for ordinary categories and calligraphic letters like $\mathcal{C}$ for locally graded categories.

We define a locally graded category of graded objects, which we use throughout the paper, and then give some further examples.

Definition 10. *Let* $\mathbb{C}$ *be an ordinary category. The locally* $\mathbb{E}$*-graded category* $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$ *is defined as follows.*

- *Objects* $X \in |\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})|$ *are* $\mathbb{E}$*-graded objects of* $\mathbb{C}$ *(Definition 1).*
- *Morphisms* $f : X -e\!\!\rightarrow Y$ *are natural transformations* $f : X \Rightarrow Y(-\cdot e)$*.*
- *The identity* id_X *is the identity natural transformation* $X \Rightarrow X$*.*
- *The composition* $g \circ f : X -e\cdot e'\!\!\rightarrow Z$ *of* $f : X -e\!\!\rightarrow Y$ *and* $g : Y -e'\!\!\rightarrow Z$ *is* $X \overset{f}{\Longrightarrow} Y(-\cdot e) \xRightarrow{g_{-\cdot e}} Z(-\cdot e \cdot e')$*.*
- *The coercion* $\zeta^* f : X -e'\!\!\rightarrow Y$ *of* $f : X -e\!\!\rightarrow Y$ *along* $\zeta \in \mathbb{E}(e, e')$ *is* $X \overset{f}{\Longrightarrow} Y(-\cdot e) \xRightarrow{Y(-\cdot\zeta)} Y(-\cdot e')$*.*

Example 1. Just as monoids in **Set** are categories with one object, $\mathbb{N}^+_\leq$-graded monoids A in **Set** are locally $\mathbb{N}^+_\leq$-graded categories with one object (morphisms of grade e are elements of Ae).

Example 2. Using both the multiplicative and additive monoidal structures on $\mathbb{N}_{\leq}$, there is a locally $\mathbb{N}^{\times}_{\leq}$-graded category **GMon** that has $\mathbb{N}^{+}_{\leq}$-graded monoids as objects. Morphisms $f : \mathsf{A} \mathrel{-e\!\!\to} \mathsf{A}'$ in **GMon** are homomorphisms $f : \mathsf{A} \to \mathsf{A}'(- \cdot e)$, where $\mathsf{A}'(- \cdot e)$ is the graded monoid $(A'(- \cdot e), u, m_{-\cdot e, -\cdot e})$ for $\mathsf{A}' = (A', u, m)$. Identities, composition, and coercions are as in $\mathbf{GObj}_{\mathbb{N}^{\times}_{\leq}}(\mathbf{Set})$.

We have similar locally graded categories for our other two examples. For a fixed $\mathbb{N}^{+}_{\leq}$-graded monoid M, the M-actions form a locally $\mathbb{N}^{+}_{\leq}$-graded category $\mathbf{GAct}_{\mathsf{M}}$, in which morphisms $\mathsf{A} \mathrel{-e\!\!\to} \mathsf{A}'$ in $\mathbf{GAct}_{\mathsf{M}}$ are homomorphisms $\mathsf{A} \to \mathsf{A}'(- + e)$, where $\mathsf{A}'(- + e) = (A'(- + e), \mathrm{act}_{-,-+e})$. The graded arithmoids form a locally $\mathbb{Z}^{+}_{\leq}$-graded category **GArith** in which morphisms $\mathsf{A} \mathrel{-e\!\!\to} \mathsf{A}'$ are similarly homomorphisms $\mathsf{A} \to \mathsf{A}'(- + e)$.

We also need to consider functors between locally graded categories, and natural transformations between these functors.

Definition 11 ([26]). *A* functor $F : \mathcal{C} \to \mathcal{D}$ *between locally graded categories consists of an object mapping* $F : |\mathcal{C}| \to |\mathcal{D}|$ *and a mapping of morphisms as on the left below; these are required to preserve identities, composition and coercion as on the right below.*

$$\frac{f : X \mathrel{-e\!\!\to} Y}{Ff : FX \mathrel{-e\!\!\to} FY} \qquad \begin{gathered} F\mathrm{id}_X = \mathrm{id}_{FX} \\ F(g \circ f) = Fg \circ Ff \\ F(\zeta^* f) = \zeta^*(Ff) \end{gathered}$$

A natural transformation $\alpha : F \Rightarrow G$ *between functors* $F, G : \mathcal{C} \to \mathcal{D}$ *consists of a morphism* $\alpha_X : FX \mathrel{-1\!\!\to} GX$ *for each* $X \in |\mathcal{C}|$*, such that* $\alpha_Y \circ Ff = Gf \circ \alpha_X$ *for every* $f : X \mathrel{-e\!\!\to} Y$*.*

We of course have identity functors $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, and functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_3$ have a composition $G \cdot F : \mathcal{C}_1 \to \mathcal{C}_3$. There are also horizontal and vertical compositions of natural transformations.

Example 3. There is a forgetful functor $\mathbf{GMon} \to \mathbf{GObj}_{\mathbb{N}^{\times}_{\leq}}(\mathbf{Set})$ that sends each graded monoid A to its carrier A, and each morphism $f : \mathsf{A} \mathrel{-e\!\!\to} \mathsf{A}'$ to itself. We similarly have forgetful functors $\mathbf{GAct}_{\mathsf{M}} \to \mathbf{GObj}_{\mathbb{N}^{+}_{\leq}}(\mathbf{Set})$ and $\mathbf{GArith} \to \mathbf{GObj}_{\mathbb{Z}^{+}_{\leq}}(\mathbf{Set})$.

If $\mathcal{A}, \mathcal{A}'$ are two locally graded categories that can similarly be equipped with forgetful functors $U : \mathcal{A} \to \mathcal{C}$ and $U' : \mathcal{A}' \to \mathcal{C}$, we say that a functor $G : \mathcal{A} \to \mathcal{A}'$ is *over* $\mathcal{C}$ when $U' \cdot G = U$, i.e. when G preserves carriers and sends morphisms to themselves. For example, since addition of natural numbers is commutative, there is a functor $\mathbf{GMon} \to \mathbf{GMon}$ over $\mathbf{GObj}_{\mathbb{N}^{\times}_{\leq}}(\mathbf{Set})$ that swaps the arguments of the multiplication of each graded monoid.

Locally graded categories induce ordinary categories and vice versa. We use these constructions in our formulation of graded monads in terms of locally graded categories.

Definition 12. *Every locally graded category $\mathcal{C}$ has an* underlying *ordinary category $\underline{\mathcal{C}}$ with the same objects; morphisms $f : X \to Y$ in $\underline{\mathcal{C}}$ are morphisms $f : X \overset{1}{\rightarrow} Y$ in $\mathcal{C}$, and these compose as in $\mathcal{C}$. Every functor $F : \mathcal{C} \to \mathcal{D}$ between locally graded categories restricts to an ordinary functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$.*

In the other direction, every ordinary category $\mathbb{C}$ induces a locally $\mathbb{E}$-graded category $\mathbf{Free}_{\mathbb{E}}(\mathbb{C})$, defined by:

$$|\mathbf{Free}_{\mathbb{E}}(\mathbb{C})| = |\mathbb{C}| \qquad \mathbf{Free}_{\mathbb{E}}(\mathbb{C})(X,Y)e = \mathbb{E}(1,e) \times \mathbb{C}(X,Y)$$
$$\mathrm{id}_X = (\mathrm{id}_1, \mathrm{id}_X) \qquad (\xi', g) \circ (\xi, f) = (\xi \cdot \xi', g \circ f) \qquad \zeta^*(\xi, f) = (\zeta \circ \xi, f)$$

$\mathbf{Free}_{\mathbb{E}}(\mathbb{C})$ is free on $\mathbb{C}$ in the following sense. Let $H_{\mathbb{C}} : \mathbb{C} \to \underline{\mathbf{Free}_{\mathbb{E}}(\mathbb{C})}$ be the ordinary functor defined on objects by $H_{\mathbb{C}}X = X$ and on morphisms by $H_{\mathbb{C}}f = (\mathrm{id}_1, f)$. Then every ordinary functor $F : \mathbb{C} \to \underline{\mathcal{D}}$ induces a unique $F^\sharp : \mathbf{Free}_{\mathbb{E}}(\mathbb{C}) \to \mathcal{D}$ such that $\underline{F^\sharp} \cdot H_{\mathbb{C}} = F$; this is given on objects by $F^\sharp X = FX$, and on morphisms $(\xi, f) : X \overset{e}{\rightarrow} Y$ by $F^\sharp(\xi, f) = \xi^*(Ff) : FX \overset{e}{\rightarrow} FY$.

We can view $\mathbf{Free}_{\mathbb{E}}(\mathbb{C})$ as a full sub-locally graded category of $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$ as follows, assuming $\mathbb{C}$ has enough coproducts. Let X be an object of $\mathbb{C}$ (equivalently, an object of $\mathbf{Free}_{\mathbb{E}}(\mathbb{C})$). For each set A, we write $A \bullet X$ for the coproduct of A-many copies of X, if it exists. (This is the copower of X by A.) In particular, if $\mathbb{E}(1,e) \bullet X$ exists for every $e \in |\mathbb{E}|$, then we have a graded object $J_{\mathbb{C}}X = \mathbb{E}(1,-) \bullet X \in |\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})|$; in this way, we can view every object X of $\mathbf{Free}_{\mathbb{E}}(\mathbb{C})$ as an object $J_{\mathbb{C}}X$ of $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$. By the Yoneda lemma, morphisms $J_{\mathbb{C}}X \overset{e}{\rightarrow} Y$ in $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$ are in bijection with morphisms $X \to Ye$ in $\mathbb{C}$:

$$\dfrac{\dfrac{\mathbb{E}(1,-) \bullet X \Rightarrow Y(- \cdot e) \text{ in } [\mathbb{E}, \mathbb{C}]}{\mathbb{E}(1,-) \Rightarrow \mathbb{C}(X, Y(- \cdot e)) \text{ in } [\mathbb{E}, \mathbf{Set}]}}{X \to Ye \text{ in } \mathbb{C}}$$

Intuitively, $J_{\mathbb{C}}X$ can be thought of as the graded object generated by assigning the grade 1 to each element of X.

Definition 13. *Let $\mathbb{C}$ be an ordinary category with coproducts of the form $\mathbb{E}(1,e) \bullet X$. We define $J_{\mathbb{C}} : \mathbf{Free}_{\mathbb{E}}(\mathbb{C}) \to \mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$ to be unique such that $\underline{J_{\mathbb{C}}} \cdot H_{\mathbb{C}}$ is the ordinary functor $(X \mapsto \mathbb{E}(1,-) \bullet X) : \mathbb{C} \to [\mathbb{E}, \mathbb{C}] = \underline{\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})}$.*

Remark 1. We end this section by mentioning that, as shown by Wood [26, Theorem 1.6], locally graded category theory can be viewed as an instance of enriched category theory. Enriched category theory provides a useful source of concepts and results for grading; for example, the definition of the underlying ordinary category $\underline{\mathcal{C}}$ is just an instance of the more general definition of the underlying category of an enriched category (cf. [11, Section 1.3]). In more detail, $[\mathbb{E}, \mathbf{Set}]$ forms a monoidal category $([\mathbb{E}, \mathbf{Set}], I, \otimes)$ with *Day convolution*:

$$I = \mathbb{E}(1,-) \qquad X \otimes Y = \int^{e_1, e_2 \in \mathbb{E}} \mathbb{E}(e_1 \cdot e_2, -) \times Xe_1 \times Ye_2$$

Locally graded categories $\mathcal{C}$ are $[\mathbb{E}, \mathbf{Set}]$-categories, with coercions making $\mathcal{C}(X, Y)$ into an object of $[\mathbb{E}, \mathbf{Set}]$, and identities and composition in $\mathcal{C}$ providing identity and composition morphisms in $[\mathbb{E}, \mathbf{Set}]$ (with composition in diagram order). Functors between locally graded categories are $[\mathbb{E}, \mathbf{Set}]$-functors between $[\mathbb{E}, \mathbf{Set}]$-categories, and similarly for natural transformations, so that the 2-categories of locally graded categories and of $[\mathbb{E}, \mathbf{Set}]$-categories are equivalent.

4 Flexibly and Rigidly Graded Monads

We next employ locally graded categories to define notions of flexibly graded monad and rigidly graded monad. Rigidly graded monads turn out to be exactly graded monads (Definition 2); we say 'rigidly' to more clearly distinguish between these and flexibly graded monads. Flexibly graded monads are intuitively more general than rigidly graded monads. There is a flexibly graded monad whose algebras are graded monoids, and one whose algebras are graded arithmoids.

Both notions arise as instances of the following definition of *relative monad* for locally graded categories. Relative monads are similar to monads, except that instead of having free algebras on every object, they only have free algebras on objects of the form JX where $J : \mathcal{J} \to \mathcal{C}$ is some functor (which should be thought of as a full sub-locally graded category of $\mathcal{C}$; every J we use below is fully faithful in the sense that the functions $(f \mapsto Jf) : \mathcal{J}(X, Y)e \to \mathcal{C}(JX, JY)e$ are bijective). Altenkirch et al. [1] give a definition of relative monad for ordinary categories; their definition generalizes easily to enriched categories (cf. Staton [23]), and the definition we give below arises from this via the discussion in Remark 1.

Definition 14. *Let $J : \mathcal{J} \to \mathcal{C}$ be a functor between locally graded categories. A* J-relative monad T *consists of an object mapping $T : |\mathcal{J}| \to |\mathcal{C}|$, a* unit *$\eta_X : JX \mathrel{-1\!\!\to} TX$ for each $X \in |\mathcal{J}|$, and a* Kleisli extension *operator*

$$\frac{f : JX \mathrel{-e\!\!\to} TY}{f^\dagger : TX \mathrel{-e\!\!\to} TY}$$

which is required to be unital, associative, and natural, as follows:

$$\begin{array}{rcll} f^\dagger \circ \eta_X &=& f & \text{for each } f : JX \mathrel{-e\!\!\to} TY \\ \mathrm{id}_{TX} &=& \eta_X^\dagger & \text{for each } X \in |\mathcal{J}| \\ (g^\dagger \circ f)^\dagger &=& g^\dagger \circ f^\dagger & \text{for each } f : JX \mathrel{-e\!\!\to} TY, g : JY \mathrel{-e'\!\!\to} TZ \\ (\zeta^* f)^\dagger &=& \zeta^*(f^\dagger) & \text{for each } \zeta \in \mathbb{E}(e, e'), f : JX \mathrel{-e\!\!\to} TY \end{array}$$

A morphism *$\alpha : \mathsf{T} \to \mathsf{T}'$ of J-relative monads is a family of morphisms $\alpha_X : TX \mathrel{-1\!\!\to} T'X$ in $\mathcal{C}$ such that $\alpha_X \circ \eta_X = \eta_X$ for all $X \in |\mathcal{J}|$ and such that $(\alpha_Y \circ f)^\dagger \circ \alpha_X = \alpha_Y \circ f^\dagger$ for all $f : JX \mathrel{-e\!\!\to} TY$.*

The object mapping of each J-relative monad T extends to a functor $T : \mathcal{J} \to \mathcal{C}$ by defining $Tf = (\eta_Y \circ Jf)^\dagger$ for each $f : X \mathrel{-e\!\!\to} Y$; under this definition, units, Kleisli extensions, and morphisms of relative monads are natural in the appropriate sense. The following two instances of the above definition are the ones that matter for us. Here $J_{\mathbb{C}}$ is as in Definition 13.

Definition 15. *Let* $\mathbb{C}$ *be an ordinary category with coproducts of the form* $\mathbb{E}(1,e) \bullet X$*. A* flexibly $\mathbb{E}$-graded monad on $\mathbb{C}$ *is a monad on the locally graded category* $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$*, i.e. an* $\mathrm{Id}_{\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})}$*-relative monad. A* rigidly $\mathbb{E}$-graded monad on $\mathbb{C}$ *is a* $J_{\mathbb{C}}$*-relative monad.*

We now prove our claim that graded monads can be formulated in terms of locally graded categories, by showing that they are just rigidly graded monads in the sense of the above definition. The following table compares the data the two definitions (Definition 2, Definition 15) ask for.

	Graded monad T	Rigidly graded monad T
Object mapping	$T : \lvert\mathbb{C}\rvert \to \lvert[\mathbb{E}, \mathbb{C}]\rvert$	$T : \lvert\mathbf{Free}_{\mathbb{E}}(\mathbb{C})\rvert \to \lvert\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})\rvert$
Unit	$\eta_X : X \to TX1$	$\eta_X : \mathbb{E}(1,-) \bullet X \Rightarrow TX$
Kleisli extension	$\dfrac{f : X \to TYe}{f^\dagger : TX \Rightarrow TY(- \cdot e)}$	$\dfrac{f : \mathbb{E}(1,-) \bullet X \Rightarrow TY(- \cdot e)}{f^\dagger : TX \Rightarrow TY(- \cdot e)}$

The object mappings have identical types (since $\lvert\mathbf{Free}_{\mathbb{E}}(\mathbb{C})\rvert = \lvert\mathbb{C}\rvert$ and $\lvert\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})\rvert = \lvert[\mathbb{E}, \mathbb{C}]\rvert$). The units and Kleisli extensions do not have identical types, but are in bijection via the Yoneda lemma (morphisms $X \to Ye$ are in bijection with natural transformations $\mathbb{E}(1,-) \bullet X \Rightarrow Y(- \cdot e)$).

Theorem 1. *There is a bijection between* $\mathbb{E}$*-graded monads on* $\mathbb{C}$ *and rigidly* $\mathbb{E}$*-graded monads on* $\mathbb{C}$ *for each* $\mathbb{C}$ *with coproducts of the form* $\mathbb{E}(1,e) \bullet X$*.*

From this point onwards, we view List, Wr^{M} and Count as rigidly graded monads.

Remark 2. We can also consider $K_{\mathbb{C}}$-relative monads, where $K_{\mathbb{C}} : \mathbb{C} \to [\mathbb{E}, \mathbb{C}]$ is the functor between ordinary categories defined by $K_{\mathbb{C}}X = \mathbb{E}(1,-) \bullet X$. These are similar to graded monads, but not the same. The Kleisli extension of a $K_{\mathbb{C}}$-relative monad has the form on the right below (equivalently, the form on the left below).

$$\frac{f : X \to TY1}{f^\dagger : TX \Rightarrow TY} \qquad \frac{f : \mathbb{E}(1,-) \bullet X \Rightarrow TY}{f^\dagger : TX \Rightarrow TY}$$

Compared to the table above, this is missing the quantification over e. The quantification over e is what locally graded categories (as opposed to ordinary categories) provide. This is why ordinary categories do not suffice when working with graded monads, and why we instead consider locally graded categories. (where e appears as the grade of a morphism). (Fujii et al. [4] instead use categories equipped with an action of $\mathbb{E}$; the grade e then appears when applying the action. We discuss this approach further in Section 4.2 below.)

Turning to flexibly graded monads, their data are the following:

$$T : |\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})| \to |\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})|$$

$$\eta_X : X \Rightarrow TX$$

$$\frac{f : X \Rightarrow TY(- \cdot e)}{f^\dagger : TX \Rightarrow TY(- \cdot e)}$$

We use the following examples.

Example 4. We define a flexibly $\mathbb{N}^{\times}_{\leq}$-graded monad $\mathsf{List}_{\mathrm{flex}}$ on **Set**, whose algebras are $\mathbb{N}^{+}_{\leq}$-graded monoids (Theorem 2 below). Informally, $\mathrm{List}_{\mathrm{flex}}Xe$ is the set of lists over $X : \mathbb{N}_{\leq} \to \mathbf{Set}$ whose total grade is at most $e \in \mathbb{N}$. To define $\mathsf{List}_{\mathrm{flex}}$ formally, let S_e be the poset of lists $\vec{n} = (n_1, \ldots, n_k)$ of natural numbers whose sum is at most $e \in \mathbb{N}$. (These lists may be empty, and any number of elements may be 0.) The ordering is pointwise, i.e. $\vec{n} \leq \vec{n}'$ if $\vec{n}$ and $\vec{n}'$ have the same length and $n_i \leq n'_i$ for all i. Then for each graded object $X : \mathbb{N}_{\leq} \to \mathbf{Set}$, we define a graded object $\mathrm{List}_{\mathrm{flex}}X : \mathbb{N}_{\leq} \to \mathbf{Set}$ by

$$\mathrm{List}_{\mathrm{flex}}Xe = \mathrm{colim}_{\vec{n} \in S_e} \textstyle\prod_i Xn_i \qquad \mathrm{List}_{\mathrm{flex}}X(e{\leq}e') = [\mathrm{in}_{\vec{n}}]_{\vec{n} \in S_e}$$

(Recall that we write $e{\leq}e'$ for the unique element of $\mathbb{N}_{\leq}(e, e')$; we also write in_i for the ith coprojection of a colimit.) Here we use the fact that if $e \leq e'$ then $S_e \subseteq S_{e'}$. For the unit $\eta_X : X \mathrel{-1\!\!\to} \mathrm{List}_{\mathrm{flex}}X$ (i.e. $\eta_X : X \Rightarrow \mathrm{List}_{\mathrm{flex}}X$), we use singleton lists, defining $\eta_{X,d}x = \mathrm{in}_{(d)}(x)$. Given $f : X \mathrel{-e\!\!\to} \mathrm{List}_{\mathrm{flex}}Y$ in $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$ (i.e. $f : X \Rightarrow \mathrm{List}_{\mathrm{flex}}Y(- \cdot e)$), the Kleisli extension $f^\dagger : \mathrm{List}_{\mathrm{flex}}X \Rightarrow \mathrm{List}_{\mathrm{flex}}Y(- \cdot e)$ is defined by

$$f^\dagger_d(\mathrm{in}_{\vec{n}}(x_1, \ldots, x_k)) = \mathrm{in}_{\vec{m}_1 \cdots \vec{m}_k}(y_{11}, \ldots, y_{1\ell_1}, \ldots, y_{k1}, \ldots, y_{k\ell_k})$$
$$\text{where } \mathrm{in}_{\vec{m}_i}(y_{i1}, \ldots, y_{i\ell_i}) = f_{n_i}x_i$$

Here we use the fact that, if the sum of $\vec{n}$ is at most d, and the sum of each $\vec{m}_i$ is at most $n_i \cdot e$, then the sum of the concatenation $\vec{m}_1 \cdots \vec{m}_k$ is at most $\sum_i (n_i \cdot e) = (\sum_i n_i) \cdot e \leq d \cdot e$. Informally, $f^\dagger$ takes a list, applies f to each element, and then concatenates the results.

Example 5. Let M be a $\mathbb{N}^{+}_{\leq}$-graded monoid. There is a flexibly $\mathbb{N}^{+}_{\leq}$-graded writer monad $\mathsf{Wr}^{\mathsf{M}}_{\mathrm{flex}}$ defined on objects by $\mathsf{Wr}^{\mathsf{M}}_{\mathrm{flex}}X = M \otimes X$ where $\otimes$ is Day convolution (see Remark 1). This turns out to have the same algebras (namely M-actions) as the rigidly graded monad Wr^{M} (see Example 7 below), in contrast to the situation with $\mathsf{List}_{\mathrm{flex}}$ and List.

Example 6. We have a flexibly $\mathbb{Z}^{+}_{\leq}$-graded monad $\mathsf{Count}_{\mathrm{flex}}$ whose algebras are graded arithmoids (Theorem 3 below). For each graded object $X : \mathbb{Z}_{\leq} \to \mathbf{Set}$, the graded object $\mathrm{Count}_{\mathrm{flex}}X : \mathbb{Z}_{\leq} \to \mathbf{Set}$ is defined by

$$\mathrm{Count}_{\mathrm{flex}}Xe = \{t : \textstyle\prod_{i:\mathbb{N}} \coprod_{j:\mathbb{N}} X(e - (j - i)) \mid$$
$$\exists \rho \in \mathbb{N}.\, \forall k, j \in \mathbb{N}, x.\, t\,\rho = (j, x) \Rightarrow t(\rho + k) = (j + k, x)\}$$

The intuition is similar to that of Count above: a computation t takes an initial counter value i and returns a pair (j, x) of a final counter value j and result x. Note however that here the increase $j - i$ in the value of the counter may be greater than e; this is "corrected" by the fact that the grade of x is then negative. The unit and Kleisli extension are similar to those of Count:

$$\eta_{X,d} x = \lambda i.\,(i, x) \qquad f_d^{\dagger} t = \lambda i.\,\mathrm{let}\,(j, x) = t\,i\ \mathrm{in}\ f_{d-(j-i)}\,x\,j$$

4.1 Eilenberg-Moore and Kleisli

Every relative monad T induces a locally graded category $\mathbf{EM}(\mathsf{T})$ of (Eilenberg-Moore) T-*algebras*, which is analogous to the usual Eilenberg-Moore category of a monad. We define $\mathbf{EM}(\mathsf{T})$, and prove a few basic properties; as for the definition of relative monad, these come directly from considering relative monads in enriched categories more generally (via Remark 1).

Definition 16. *Let* T *be a* J*-relative monad for some functor* $J : \mathcal{J} \to \mathcal{C}$ *between locally graded categories. A* T-algebra $\mathsf{A} = (A, (-)^{\ddagger})$ *is a pair of a* carrier $A \in |\mathcal{C}|$ *and an* extension *operator*

$$\frac{f : JX \xrightarrow{e} A}{f^{\ddagger} : TX \xrightarrow{e} A}$$

which is required to satisfy the following equations:

$$\begin{array}{rll} f^{\ddagger} \circ \eta_X = f & & \text{for each } f : JX \xrightarrow{e} A \\ (g^{\ddagger} \circ f)^{\ddagger} = g^{\ddagger} \circ f^{\dagger} & & \text{for each } f : JX \xrightarrow{e} TY, g : JY \xrightarrow{e'} A \\ (\zeta^* f)^{\ddagger} = \zeta^*(f^{\ddagger}) & & \text{for each } \zeta \in \mathbb{E}(e, e'), f : JX \xrightarrow{e} A \end{array}$$

These are the objects of a locally graded category $\mathbf{EM}(\mathsf{T})$. *Morphisms* $f : \mathsf{A} \xrightarrow{e} \mathsf{A}'$ *in* $\mathbf{EM}(\mathsf{T})$ *are morphisms* $f : A \xrightarrow{e} A'$ *in* $\mathcal{C}$ *such that* $f \circ g^{\ddagger} = (f \circ g)^{\ddagger}$ *for each* $g : JX \xrightarrow{e'} A$*; identities, composition and coercions are as in* $\mathcal{C}$*. The* forgetful functor $U_{\mathsf{T}} : \mathbf{EM}(\mathsf{T}) \to \mathcal{C}$ *sends* A *to* A*, and morphisms to themselves.*

We use this definition as our notion of algebra for rigidly and flexibly graded monads. Fujii et al. [4] define a notion of *graded algebra* for a graded monad T. When T is a rigidly graded monad, the T-algebras as defined as above are in bijection with graded algebras; see Section 4.2 below.

We characterize the algebras of the three flexibly graded monads defined above. First, we note that for every flexibly graded monad T, the T-algebras can be formulated equivalently as a pair of a carrier and a structure map (analogously to the standard definition of Eilenberg-Moore algebra), rather than in the extension form above. (This is an instance of a more general result, see Marmolejo and Wood [15].) The flexibly graded monad T has a *multiplication* $\mu : T \cdot T \Rightarrow T$, defined by $\mu_X = \mathrm{id}_{TX}^{\dagger}$. Each T-algebra A induces a morphism $a : TA \xrightarrow{1} A$ by $a = \mathrm{id}_A{}^{\ddagger}$, and this gives us a bijection between T-algebras A and pairs (A, a) of an object $A \in |\mathcal{C}|$ and a morphism $a : TA \xrightarrow{1} A$ compatible with the unit and multiplication of T (i.e. $a \circ \eta_A = \mathrm{id}_A$ and $a \circ \mu_A = a \circ Ta$).

As our first example, the flexibly $\mathbb{N}^{\times}_{\leq}$-graded monad $\mathsf{List}_{\mathrm{flex}}$ has graded monoids as algebras.

Theorem 2. *There exists an isomorphism* $\mathbf{GMon} \cong \mathbf{EM}(\mathsf{List}_{\mathrm{flex}})$ *over* $\mathbf{GObj}_{\mathbb{N}^{\times}_{\leq}}(\mathbf{Set})$*.*

Proof. Algebras A of $\mathsf{List}_{\mathrm{flex}}$ are, as above, in bijection with pairs (A, a) of a graded object A and morphism $a : \mathrm{List}_{\mathrm{flex}} A \xrightarrow{-1} A$ (i.e. natural transformation $a : \mathrm{List}_{\mathrm{flex}} A \Rightarrow A$) compatible with the unit and multiplication of $\mathsf{List}_{\mathrm{flex}}$. Given (A, a), we define the multiplication and unit of a graded monoid (A, u, m) by

$$u = a_0(\mathrm{in}_{()}()) \qquad m_{e_1,e_2}(x_1, x_2) = a_{e_1+e_2}(\mathrm{in}_{(e_1,e_2)}(x_1, x_2))$$

In the other direction, given a graded monoid (A, u, m), we define a as follows (omitting the subscripts of m):

$$a(\mathrm{in}_{(n_1,\ldots,n_k)}(x_1, \ldots, x_k)) = m(x_1, m(x_2, \cdots m(x_{k-1}, m(x_k, u)) \cdots))$$

Simple calculations show that these form a bijection between $\mathsf{List}_{\mathrm{flex}}$-algebras and graded monoids, and that a morphism $A \xrightarrow{-e} A'$ in $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$ is a morphism of algebras if and only if it is a morphism of the corresponding graded monoids.

As an example of this theorem, the free algebra $F_{\mathsf{List}_{\mathrm{flex}}} X$ forms the free graded monoid on $X : \mathbb{N}_{\leq} \to \mathbf{Set}$, with unit $u = \mathrm{in}_{()}()$ and multiplication

$$m_{d,e}(\mathrm{in}_{\vec{n}}(x_1, \ldots, x_k), \mathrm{in}_{\vec{m}}(x'_1, \ldots, x'_\ell)) = \mathrm{in}_{\vec{n}\vec{m}}(x_1, \ldots, x_k, x'_1, \ldots, x'_\ell)$$

Example 7. As our second example, the rigidly graded and flexibly graded writer monads have the same algebras: in both cases the algebras are M-actions, and there are isomorphisms $\mathbf{EM}(\mathsf{Wr}^{\mathsf{M}}_{\mathrm{flex}}) \cong \mathbf{GAct}_{\mathsf{M}} \cong \mathbf{EM}(\mathsf{Wr}^{\mathsf{M}})$ over $\mathbf{GObj}_{\mathbb{N}^{+}_{\leq}}(\mathbf{Set})$. An algebra for the flexibly graded monad $\mathsf{Wr}^{\mathsf{M}}_{\mathrm{flex}}$ is (as above) equivalently a pair of a graded object $A : \mathbb{N}_{\leq} \to \mathbf{Set}$ and a natural transformation $a : \mathsf{Wr}^{\mathsf{M}}_{\mathrm{flex}} A \Rightarrow A$ compatible with the unit and multiplication. These are equivalently M-actions by properties of Day convolution. For the rigidly graded monad Wr^{M}, algebras are again in bijection with M-actions, in particular, for each Wr^{M}-algebra A, we have functions $[A(\zeta + e_2)]_{\zeta \in \mathbb{N}_{\leq}(0,d)} : J_{\mathbf{Set}}(Ae_2)d \to A(d + e_2)$, and using these the graded monoid M acts on the carrier of A with $\mathrm{act}_{e_1,e_2} = ([A(\zeta + e_2)]_{\zeta})^{\ddagger}_{e_1}$.

Finally, for our third example we show that the flexibly $\mathbb{Z}^{+}_{\leq}$-graded monad $\mathsf{Count}_{\mathrm{flex}}$ has graded arithmoids as algebras.

Theorem 3. *There exists an isomorphism* $\mathbf{GArith} \cong \mathbf{EM}(\mathsf{Count}_{\mathrm{flex}})$ *over* $\mathbf{GObj}_{\mathbb{Z}^{+}_{\leq}}(\mathbf{Set})$*.*

Proof. Each algebra A of $\mathsf{Count}_{\mathrm{flex}}$ comes with a natural transformation $a : \mathrm{Count}_{\mathrm{flex}} A \Rightarrow A$, and forms a graded arithmoid by defining

$$\mathrm{inc}_e x = a_{e+1}(\lambda i.\, (i + 1, x))$$
$$\mathrm{dec}_e(x, y) = a_e(\lambda i.\, \text{if } i = 0 \text{ then } (0, x) \text{ else } (i - 1, y))$$

Conversely, to make a graded arithmoid into a $\mathsf{Count}_{\mathrm{flex}}$-algebra, we define $a : \mathrm{Count}_{\mathrm{flex}} A \Rightarrow A$ as follows. Let $\mathrm{inc}^j_e : Ae \to A(e+j)$ be given by composing inc with itself j times. Given $t \in \mathrm{Count}_{\mathrm{flex}} Ae$, let ρ be a witness to the side-condition in the definition of $\mathrm{Count}_{\mathrm{flex}} Ae$, and set $(j_i, x_i) = t\, i$. We then define

$$a_e t = \mathrm{dec}(\mathrm{inc}^{j_0} x_0, \mathrm{dec}(\mathrm{inc}^{j_1} x_1, \cdots (\mathrm{dec}(\mathrm{inc}^{j_{\rho-1}} x_{\rho-1}, \mathrm{inc}^{j_\rho} x_\rho)) \cdots))$$

(Here it does not matter which witness ρ is chosen because of the graded arithmoid law $\mathrm{dec}_e(x, \mathrm{inc}_e x) = x$. We can take for example the smallest such ρ.)

We frequently look at relative monads in terms of their algebras; this is justified by the fact that each relative monad is completely determined by its algebras. For example, $\mathsf{List}_{\mathrm{flex}}$ is (up to isomorphism) the only flexibly graded monad that has graded monoids as algebras. To make this precise, if $\alpha : \mathsf{T}' \to \mathsf{T}$ is a morphism of J-relative monads, then we let $\mathbf{EM}(\alpha) : \mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\mathsf{T}')$ be the functor over $\mathcal{C}$ that sends $(A, (-)^\ddagger)$ to $(A, (-)^{\ddagger'})$ where $f^{\ddagger'} = f^\ddagger \circ \alpha$. The following is a general fact about relative monads, specialized to locally graded categories:

Lemma 1. *Let* T *and* T' *be* J*-relative monads where* $J : \mathcal{J} \to \mathcal{C}$*. For every functor* $G : \mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\mathsf{T}')$ *over* $\mathcal{C}$*, there is a unique relative monad morphism* $\alpha : \mathsf{T}' \to \mathsf{T}$ *such that* $\mathbf{EM}(\alpha) = G$*.*

The assignment $\alpha \mapsto \mathbf{EM}(\alpha)$ is therefore a bijection between morphisms $\mathsf{T}' \to \mathsf{T}$ and functors $\mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\mathsf{T}')$ over $\mathcal{C}$. It follows that, if T' and T have the same algebras, in the sense that there exists an isomorphism $\mathbf{EM}(\mathsf{T}) \cong \mathbf{EM}(\mathsf{T}')$ over $\mathcal{C}$, then there also exists an isomorphism $\mathsf{T}' \cong \mathsf{T}$ of relative monads.

Remark 3. Lemma 1 relies on considering locally graded categories of algebras instead of the underlying ordinary categories: in general, there are ordinary functors $\underline{\mathbf{EM}(\mathsf{T})} \to \underline{\mathbf{EM}(\mathsf{T}')}$ over $\underline{\mathcal{C}}$ that are not of the form $\underline{\mathbf{EM}(\alpha)}$. In particular, there are examples of this in which T and T' are rigidly graded monads.

If T is a J-relative monad (where $J : \mathcal{J} \to \mathcal{C}$), then the *free* T*-algebra* $F_{\mathsf{T}} X$ on $X \in |\mathcal{J}|$ has TX as carrier and Kleisli extension $(-)^\dagger$ as extension operator. Since X ranges over objects of $\mathcal{J}$, these alone do not provide a left adjoint to the forgetful functor $U_{\mathsf{T}} : \mathbf{EM}(\mathsf{T}) \to \mathcal{C}$. Instead, the free algebras form the left J*-relative* adjoint $F_{\mathsf{T}} : \mathcal{J} \to \mathbf{EM}(\mathsf{T})$ of $U_{\mathsf{T}} : \mathbf{EM}(\mathsf{T}) \to \mathcal{C}$.

Definition 17 ([25]). *Let* $J : \mathcal{J} \to \mathcal{C}$ *be a functor between locally graded categories. A* J-relative adjunction *consists of functors* $L : \mathcal{J} \to \mathcal{D}$ *(the left adjoint) and* $R : \mathcal{D} \to \mathcal{C}$ *(the right adjoint), and a family of bijections*

$$\theta_{X,Y,e} : \mathcal{D}(LX, Y)e \cong \mathcal{C}(JX, RY)e$$

natural in X, Y, e *in the sense that the following hold for all* $f : LX \xrightarrow{e} Y$*:*

$$\begin{aligned} \theta_{X',Y,e'\cdot e}(f \circ Lg) &= \theta_{X,Y,e} f \circ Jg && \text{for each } g : X' \xrightarrow{e'} X \\ \theta_{X,Y',e\cdot e'}(g \circ f) &= Rg \circ \theta_{X,Y,e} f && \text{for each } g : Y \xrightarrow{e'} Y' \\ \theta_{X,Y,e'}(\zeta^* f) &= \zeta^*(\theta_{X,Y,e} f) && \text{for each } \zeta \in \mathbb{E}(e, e') \end{aligned}$$

Each J-relative adjunction induces a J-relative monad, with object mapping $X \mapsto R(LX)$. Conversely, $\mathbf{EM}(\mathsf{T})$ forms a *resolution* of T, i.e. a J-relative adjunction that induces the relative monad T. This is the terminal resolution of T, analogously to the situation with ordinary monads. In fact, many of the usual properties of monads carry over to relative monads in general and to flexibly and rigidly graded monads in particular. Each relative monad also has an initial resolution, given by the Kleisli construction.

Definition 18. *Let T be a J-relative monad, where $J : \mathcal{J} \to \mathcal{C}$. The* Kleisli *locally graded category $\mathbf{Kl}(\mathsf{T})$ of T has the same objects as $\mathcal{J}$. The morphisms $f : X -e\!\to Y$ in $\mathbf{Kl}(\mathsf{T})$ are morphisms $f : JX -e\!\to TY$ in $\mathcal{C}$, the identity on X is $\eta_X : JX -1\!\to TX$, the composition of $f : JX -e\!\to TY$ and $g : JY -e'\!\to TZ$ is $g^\dagger \circ f : JX -e \cdot e'\!\to TZ$, and coercions are as in $\mathcal{C}$.*

In the special case where T is a rigidly graded monad, $\mathbf{Kl}(\mathsf{T})$ is (isomorphic to) the Kleisli locally graded category defined by Gaboardi et al. [5].

4.2 $\mathbb{E}$-actegories

Previous work on graded monads, in particular by Fujii et al. [4], uses $\mathbb{E}$-*actegories* instead of locally $\mathbb{E}$-graded categories. We outline the connection between the two settings, and show that our locally graded categories of T-algebras are in some sense the same as Fujii et al.'s actegories of graded algebras.

A strict $\mathbb{E}$-*actegory* is an ordinary category $\mathbb{C}$ equipped with a bifunctor $* : \mathbb{E} \times \mathbb{C} \to \mathbb{C}$ that is compatible with the monoidal structure of $\mathbb{E}$ strictly, i.e. up to equality. Every strict $\mathbb{E}$-actegory $(\mathbb{C}, *)$ induces a locally $\mathbb{E}$-graded category $\Psi(\mathbb{C}, *)$: objects are the same as $\mathbb{C}$, and morphisms $X -e\!\to Y$ in $\Psi(\mathbb{C}, *)$ are morphisms $X \to e * Y$ in $\mathbb{C}$. This construction extends to a 2-functor Ψ (with appropriate notions of 1- and 2-cell between actegories), and Ψ is 2-fully faithful (see [16,6]). In this way, we can view strict actegories as a special case of locally graded categories. An example of a locally graded category that arises in this way is $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$: we can make $[\mathbb{E}, \mathbb{C}]$ into an actegory by defining $e * X = X(- \cdot e)$, and then $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$ is exactly $\Psi([\mathbb{E}, \mathbb{C}], *)$.

Eilenberg-Moore locally graded categories $\mathbf{EM}(\mathsf{T})$ of rigidly graded monads also arise in this way. The ordinary category $\underline{\mathbf{EM}(\mathsf{T})}$ forms a strict $\mathbb{E}$-actegory by assigning to each grade $e \in |\mathbb{E}|$ and T-algebra A the T-algebra $e * \mathsf{A}$ whose carrier is $A(- \cdot e)$, and whose extension operator is the restriction of that of A. The locally graded category $\Psi(\underline{\mathbf{EM}(\mathsf{T})}, *)$ is then exactly $\mathbf{EM}(\mathsf{T})$. Moreover, the actegory $(\underline{\mathbf{EM}(\mathsf{T})}, *)$ is isomorphic to Fujii et al.'s actegory of graded algebras. In this sense, the latter, viewed as a locally graded category, is just our $\mathbf{EM}(\mathsf{T})$.

Not all of the locally graded categories we define above arise in this way however: $\mathbf{Free}_{\mathbb{E}}(\mathbb{C})$ and $\mathbf{Kl}(\mathsf{T})$ do not. (Fujii et al. [4] define Kleisli actegories of graded monads; applying Ψ to these does not yield $\mathbf{Kl}(\mathsf{T})$.)

5 Rigidly Graded Monads from Flexibly Graded Monads

We turn to the relationship between flexibly and rigidly graded monads. In this section, we show that every flexibly graded monad induces a rigidly graded monad that is in some sense canonical. We use this construction to explain where the graded monoid structure on List$X$ comes from. Throughout this section, we suppose an ordinary category $\mathbb{C}$ with coproducts of the form $\mathbb{E}(1, e) \bullet X$.

Let T be a flexibly $\mathbb{E}$-graded monad on $\mathbb{C}$. The *rigidly graded restriction* $\lfloor \mathsf{T} \rfloor$ of T is the rigidly $\mathbb{E}$-graded monad $\lfloor \mathsf{T} \rfloor$ on $\mathbb{C}$ defined by restricting the structure of T to objects of $\mathbf{Free}_{\mathbb{E}}(\mathbb{C})$ (viewed as a sub-locally graded category of $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$ via the functor $J_{\mathbb{C}}$). Explicitly, $\lfloor \mathsf{T} \rfloor$ is given on objects by $\lfloor T \rfloor X = T(J_{\mathbb{C}} X)$, and the unit and Kleisli extension are restrictions of those of T. This construction is functorial: every morphism $\alpha : \mathsf{T} \to \mathsf{T}'$ of flexibly graded monads restricts to a morphism $\lfloor \alpha \rfloor : \lfloor \mathsf{T} \rfloor \to \lfloor \mathsf{T}' \rfloor$ of rigidly graded monads, so we have a functor between the ordinary categories of flexibly $\mathbb{E}$-graded monads on $\mathbb{C}$ and rigidly $\mathbb{E}$-graded monads on $\mathbb{C}$:

$$\lfloor - \rfloor : \mathbf{FGMnd}_{\mathbb{E}}(\mathbb{C}) \to \mathbf{RGMnd}_{\mathbb{E}}(\mathbb{C})$$

We also have a functor $R_{\mathsf{T}} : \mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\lfloor \mathsf{T} \rfloor)$ over $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$, which sends each T-algebra A to the $\lfloor \mathsf{T} \rfloor$-algebra $R_{\mathsf{T}}\mathsf{A}$ whose carrier is A, and whose extension operator $(-)^{\ddagger}$ is the restriction of that of A.

We record two crucial facts about $\lfloor \mathsf{T} \rfloor$. The first is that the graded objects $\lfloor T \rfloor X$ form T-algebras. More specifically, the free $\lfloor \mathsf{T} \rfloor$-algebra functor $F_{\lfloor \mathsf{T} \rfloor}$ is equal to

$$\mathbf{Free}_{\mathbb{E}}(\mathbb{C}) \xrightarrow{J_{\mathbb{C}}} \mathbf{GObj}_{\mathbb{E}}(\mathbb{C}) \xrightarrow{F_{\mathsf{T}}} \mathbf{EM}(\mathsf{T}) \xrightarrow{R_{\mathsf{T}}} \mathbf{EM}(\lfloor \mathsf{T} \rfloor)$$

so in particular, $\lfloor T \rfloor X$ is the carrier of the T-algebra $F_{\mathsf{T}}(J_{\mathbb{C}} X)$. This is where, for example, the graded monoid structure on List$X$ comes from; see Example 9 below.

The second fact is that $\lfloor \mathsf{T} \rfloor$ is canonical, in that it satisfies the universal property expressed in the following lemma. Informally, the Eilenberg-Moore resolution of $\lfloor \mathsf{T} \rfloor$ is as close as possible to the Eilenberg-Moore resolution of T. From this it follows that, if there is *any* rigidly graded monad T' with the same algebras as T, then T' is actually $\lfloor \mathsf{T} \rfloor$ (Corollary 1).

Lemma 2. *Let T be a flexibly $\mathbb{E}$-graded monad on $\mathbb{C}$. For every rigidly $\mathbb{E}$-graded monad T' on $\mathbb{C}$ and functor $R' : \mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\mathsf{T}')$ over $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$, there is a unique morphism $\alpha : \mathsf{T}' \to \lfloor \mathsf{T} \rfloor$ of rigidly graded monads such that $R' = \mathbf{EM}(\alpha) \cdot R_{\mathsf{T}}$.*

$$\begin{tikzcd} \mathbf{EM}(\mathsf{T}) \arrow[r, "R_{\mathsf{T}}"] \arrow[dr, "R'"'] & \mathbf{EM}(\lfloor \mathsf{T} \rfloor) \arrow[d, "\mathbf{EM}(\alpha)"] \\ & \mathbf{EM}(\mathsf{T}') \end{tikzcd} \qquad \begin{tikzcd} \lfloor \mathsf{T} \rfloor \\ \mathsf{T}' \arrow[u, dashed, "\alpha"] \end{tikzcd}$$

Proof. For each $X \in |\mathbb{C}|$, the T'-algebra $R'(F_{\mathsf{T}}(J_{\mathbb{C}} X))$ has carrier $\lfloor T \rfloor X = T(J_{\mathbb{C}} X)$, so ${\eta_{J_{\mathbb{C}} X}}^{\ddagger} : T'X \mathrel{-1\!\!\to} \lfloor T \rfloor X$. Commutativity of the triangle above on

$F_{\mathsf{T}}(J_{\mathbb{C}}X) \in \mathbf{EM}(\mathsf{T})$ implies $\alpha_X = {\eta_{J_{\mathbb{C}}X}}^{\ddagger}$, hence uniqueness of α. For existence, define $\alpha_X = {\eta_{J_{\mathbb{C}}X}}^{\ddagger}$.

Corollary 1. *Let T be a flexibly $\mathbb{E}$-graded monad on $\mathbb{C}$. If there exists a pair of a rigidly $\mathbb{E}$-graded monad T' on $\mathbb{C}$ and isomorphism $R' : \mathbf{EM}(\mathsf{T}) \cong \mathbf{EM}(\mathsf{T}')$ over $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$, then $R_{\mathsf{T}} : \mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\lfloor\mathsf{T}\rfloor)$ is an isomorphism, and there is an isomorphism $\mathsf{T}' \cong \lfloor\mathsf{T}\rfloor$ of rigidly graded monads.*

Proof. The functor R' induces a morphism $\alpha : \mathsf{T}' \to \lfloor\mathsf{T}\rfloor$ by Lemma 2. Another application of Lemma 2 shows that $R'^{-1} \cdot \mathbf{EM}(\alpha)$ is the inverse of R_{T}, so that R_{T} is an isomorphism. Since both R_{T} and R'^{-1} are isomorphisms, $\mathbf{EM}(\alpha)$ must be too, and then Lemma 1 implies α is an isomorphism $\mathsf{T}' \cong \lfloor\mathsf{T}\rfloor$.

Example 8. Corollary 1 implies that $\mathsf{Wr}^{\mathsf{M}} \cong \lfloor\mathsf{Wr}^{\mathsf{M}}_{\mathsf{flex}}\rfloor$, because Wr^{M} and $\mathsf{Wr}^{\mathsf{M}}_{\mathsf{flex}}$ have the same algebras (Example 7).

Example 9. The rigidly graded restriction of $\mathsf{List}_{\mathsf{flex}}$ is List. Indeed, the following defines an isomorphism $\psi : \mathsf{List} \cong \lfloor\mathsf{List}_{\mathsf{flex}}\rfloor$ of rigidly $\mathbb{N}^{\times}_{\leq}$-graded monads:

$$\begin{aligned}\psi_{X,e} : \qquad \mathrm{List}Xe &\to \mathrm{colim}_{\vec{n}\in S_e} \textstyle\prod_i \mathbb{E}(1, n_i) \bullet X \\ (x_1, \ldots, x_k) &\mapsto \mathrm{in}_{\underbrace{(1,\ldots,1)}_{k}}(\mathrm{in}_{\mathrm{id}_1} x_1, \ldots, \mathrm{in}_{\mathrm{id}_1} x_k)\end{aligned}$$

Recall from Theorem 2 that $\mathsf{List}_{\mathsf{flex}}$-algebras are graded monoids. Each graded object $\mathrm{List}X$ is isomorphic to the carrier of the free $\lfloor\mathsf{List}_{\mathsf{flex}}\rfloor$-algebra $F_{\lfloor\mathsf{List}_{\mathsf{flex}}\rfloor}X$, which as above forms a $\mathsf{List}_{\mathsf{flex}}$-algebra, and hence a graded monoid. One can calculate that this graded monoid structure is given by concatenation of lists. In summary, the graded monoid structure on $\mathrm{List}X$ arises by starting with the locally graded category $\mathbf{GMon}$ of graded monoids, constructing free graded monoids, which form the flexibly graded monad $\mathsf{List}_{\mathsf{flex}}$, and then showing that the restriction of $\mathsf{List}_{\mathsf{flex}}$ is List.

Lemma 2 provides a universal property for List. Every graded monoid induces a List-algebra via the following functor over $\mathbf{GObj}_{\mathbb{N}^{\times}_{\leq}}(\mathbf{Set})$:

$$R : \mathbf{GMon} \xrightarrow{\text{(Theorem 2)}} \mathbf{EM}(\mathsf{List}_{\mathsf{flex}}) \xrightarrow{R_{\mathsf{List}_{\mathsf{flex}}}} \mathbf{EM}(\lfloor\mathsf{List}_{\mathsf{flex}}\rfloor) \xrightarrow{\mathbf{EM}(\psi)} \mathbf{EM}(\mathsf{List})$$

For every rigidly $\mathbb{N}^{\times}_{\leq}$-graded monad T' and functor $R' : \mathbf{GMon} \to \mathbf{EM}(\mathsf{T}')$ over $\mathbf{GObj}_{\mathbb{N}^{\times}_{\leq}}(\mathbf{Set})$, there is a unique morphism $\alpha : \mathsf{T}' \to \mathsf{List}$ of rigidly graded monads such that $R' = \mathbf{EM}(\alpha) \cdot R$. Hence, while no rigidly graded monad has graded monoids as algebras (Theorem 4 below), List is as close as we can get.

Example 10. We have $\lfloor\mathsf{Count}_{\mathsf{flex}}\rfloor \cong \mathsf{Count}$. To see this, note that $J_{\mathbf{Set}}Xd = \emptyset$ for negative d, and $J_{\mathbf{Set}}Xd \cong X$ otherwise. Hence, if $\lambda i.\,(j_i, x_i) \in \prod_{i:\mathbb{N}} \coprod_{j:\mathbb{N}} J_{\mathbf{Set}}X(e - (j - i))$, then, for each i, we must have $e - (j_i - i) \geq 0$, so $j_i \in [0..i + e]$.

This fact has analogous consequences to the list example above. It provides an explanation for where the graded arithmoid structure of the graded object $\mathrm{Count}X$ comes from. We also obtain a functor $\mathbf{GArith} \to \mathbf{EM}(\mathsf{Count})$ that provides a universal property for Count in terms of graded arithmoids.

6 Flexibly Graded Monads from Rigidly Graded Monads

We also consider going in the opposite direction: constructing a flexibly graded monad $\lceil\mathsf{T}\rceil$ from a given rigidly graded monad T. Ideally, we would like to construct $\lceil\mathsf{T}\rceil$ so that it has the same algebras as T. (This uniquely identifies $\lceil\mathsf{T}\rceil$ up to isomorphism by Lemma 1.) In general, there does not exist a $\lceil\mathsf{T}\rceil$ with this property, but we show below that there often does, by reducing existence of $\lceil\mathsf{T}\rceil$ to existence of certain colimits. Modulo existence of these colimits, flexibly graded monads are therefore more general than rigidly graded monads.

Throughout this section, we again suppose an ordinary category $\mathbb{C}$ with coproducts of the form $\mathbb{E}(1, e) \bullet X$.

Definition 19. *If it exists, the* flexibly graded extension *of a rigidly $\mathbb{E}$-graded monad T on $\mathbb{C}$ is a flexibly $\mathbb{E}$-graded monad $\lceil\mathsf{T}\rceil$ on $\mathbb{C}$ equipped with an isomorphism $Q_{\mathsf{T}} : \mathbf{EM}(\lceil\mathsf{T}\rceil) \cong \mathbf{EM}(\mathsf{T})$ over $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$.*

Example 11. The flexibly graded extension of Wr^{M} is $\mathsf{Wr}^{\mathsf{M}}_{\mathrm{flex}}$. The isomorphism $Q_{\mathsf{Wr}^{\mathsf{M}}} : \mathbf{EM}(\mathsf{Wr}^{\mathsf{M}}_{\mathrm{flex}}) \cong \mathbf{EM}(\mathsf{Wr}^{\mathsf{M}})$ is defined in Example 7.

A basic result is that the rigidly graded restriction $\lfloor\lceil\mathsf{T}\rceil\rfloor$ is T itself, and $R_{\lceil\mathsf{T}\rceil} : \mathbf{EM}(\lceil\mathsf{T}\rceil) \to \mathbf{EM}(\lfloor\lceil\mathsf{T}\rceil\rfloor)$ is an isomorphism; this is immediate from Corollary 1. We show that, if $\lceil\mathsf{T}\rceil$ exists, then it is the free flexibly graded monad on T (Lemma 3 below). Existence of $\lceil\mathsf{T}\rceil$ for all T would imply that extensions would form an ordinary functor $\lceil-\rceil : \mathbf{RGMnd}_{\mathbb{E}}(\mathbb{C}) \to \mathbf{FGMnd}_{\mathbb{E}}(\mathbb{C})$ that is left adjoint to $\lfloor-\rfloor : \mathbf{FGMnd}_{\mathbb{E}}(\mathbb{C}) \to \mathbf{RGMnd}_{\mathbb{E}}(\mathbb{C})$. Moreover, since ϕ_{T} in the following lemma is an isomorphism, $\lceil-\rceil$ would then make $\mathbf{RGMnd}_{\mathbb{E}}(\mathbb{C})$ into a coreflective subcategory of $\mathbf{FGMnd}_{\mathbb{E}}(\mathbb{C})$.

Lemma 3. *Let T be a rigidly $\mathbb{E}$-graded monad on $\mathbb{C}$ that has a flexibly graded extension $\lceil\mathsf{T}\rceil$. There is a unique morphism $\phi_{\mathsf{T}} : \mathsf{T} \to \lfloor\lceil\mathsf{T}\rceil\rfloor$ of rigidly graded monads such that $\mathbf{EM}(\phi_{\mathsf{T}}) \cdot R_{\lceil\mathsf{T}\rceil} = Q_{\mathsf{T}}$. The unique ϕ_{T} is an isomorphism, and witnesses $\lceil\mathsf{T}\rceil$ as the free flexibly graded monad on T (with respect to $\lfloor-\rfloor$).*

Proof. By Corollary 1, the functor $R_{\lceil\mathsf{T}\rceil} : \mathbf{EM}(\lceil\mathsf{T}\rceil) \to \mathbf{EM}(\lfloor\lceil\mathsf{T}\rceil\rfloor)$ is an isomorphism, so Lemma 1 implies that there is a unique ϕ_{T} such that $\mathbf{EM}(\phi_{\mathsf{T}}) = Q_{\mathsf{T}} \cdot R^{-1}_{\lceil\mathsf{T}\rceil}$, and that ϕ_{T} is an isomorphism. To show that $\lceil\mathsf{T}\rceil$ is free, suppose a flexibly $\mathbb{E}$-graded monad T' on $\mathbb{C}$ and morphism $\psi : \mathsf{T} \to \lfloor\mathsf{T}'\rfloor$ of rigidly graded monads. We show that there is a unique morphism $\hat{\psi} : \lceil\mathsf{T}\rceil \to \mathsf{T}'$ of flexibly graded monads such that $\psi = \lfloor\hat{\psi}\rfloor \circ \phi_{\mathsf{T}}$. For every morphism $\hat{\psi} : \lceil\mathsf{T}\rceil \to \mathsf{T}'$ of flexibly graded monads we have

$$
\begin{array}{lll}
\psi = \lfloor\hat{\psi}\rfloor \circ \phi_{\mathsf{T}} & & \\
\quad\Leftrightarrow & \psi \circ \phi_{\mathsf{T}}^{-1} = \lfloor\hat{\psi}\rfloor & \phi_{\mathsf{T}} \text{ is an isomorphism} \\
\quad\Leftrightarrow & \mathbf{EM}(\phi_{\mathsf{T}}^{-1}) \cdot \mathbf{EM}(\psi) = \mathbf{EM}(\lfloor\hat{\psi}\rfloor) & \text{Lemma 1} \\
\quad\Leftrightarrow & \mathbf{EM}(\phi_{\mathsf{T}}^{-1}) \cdot \mathbf{EM}(\psi) \cdot R_{\mathsf{T}'} = R_{\lceil\mathsf{T}\rceil} \cdot \mathbf{EM}(\hat{\psi}) & \text{Lemma 2} \\
\quad\Leftrightarrow & Q_{\mathsf{T}}^{-1} \cdot \mathbf{EM}(\psi) \cdot R_{\mathsf{T}'} = \mathbf{EM}(\hat{\psi}) & R_{\lceil\mathsf{T}\rceil} \text{ is an isomorphism}
\end{array}
$$

So the result follows from Lemma 1, using the last line to define $\hat{\psi}$.

Example 12. The rigidly $\mathbb{N}^{\times}_{\leq}$-graded monad List on **Set** has a flexibly graded extension $\lceil\mathsf{List}\rceil$, which can be constructed as follows. Recall from Example 4 that S_e is the poset of lists of natural numbers that sum to at most $e \in \mathbb{N}$. We define a family of full subposets $S'_e \subseteq S_e$ inductively by three rules: $(e) \in S'_e$; if $\vec{n}_1, \ldots, \vec{n}_k \in S'_e$ for $k \geq 0$, then the concatenation $\vec{n}_1\vec{n}_2 \cdots \vec{n}_k$ is in $S'_{k \cdot e}$; and if $\vec{k} \in S'_e$ and $e \leq e'$, then $\vec{k} \in S'_{e'}$. For example, $(2, 1, 1) \in S'_4$ but $(3, 1) \notin S'_4$. Then $\lceil\mathsf{List}\rceil$ is defined in exactly the same way as $\mathsf{List}_{\text{flex}}$ (Example 4), except with S' instead of S. In particular, $\lceil\text{List}\rceil Xe = \text{colim}_{\vec{n} \in S'_e} \prod_i Xn_i$. The unit, Kleisli extension, and functoriality of $\lceil\mathsf{List}\rceil$ are well-defined because of the three rules that define S'. The isomorphism $Q_{\mathsf{List}} : \mathbf{EM}(\lceil\mathsf{List}\rceil) \to \mathbf{EM}(\mathsf{List})$ sends a $\lceil\mathsf{List}\rceil$-algebra $(A, (-)^{\ddagger})$ to the List-algebra $(A, (-)^{\ddagger'})$, where $f^{\ddagger'} : \text{List}X \mathrel{-e\!\!\to} A$ is defined for $f : J_{\mathbb{C}}X \mathrel{-e\!\!\to} A$ by

$$f^{\ddagger'}(x_1, \ldots, x_k) = f^{\ddagger}(\text{in}_{(1,\ldots,1)}(\text{in}_{\text{id}_1} x_1, \ldots, \text{in}_{\text{id}_1} x_k))$$

The inclusions $S'_e \subseteq S_e$ induce a morphism $\alpha : \lceil\mathsf{List}\rceil \to \mathsf{List}_{\text{flex}}$ of flexibly graded monads. This is not an isomorphism. For example, let $N : \mathbb{N}_{\leq} \to \mathbf{Set}$ be the graded object $Nn = \{0, \ldots, n\}$, where $N(n{\leq}n')$ is the inclusion $Nn \subseteq Nn'$. Then $\text{in}_{(3,1)}(3, 1) \in \text{List}_{\text{flex}}N4$ is not in the image of $\alpha_{N,4}$, so $\alpha_{N,4}$ is not a bijection. In fact, there is no isomorphism $\lceil\mathsf{List}\rceil \cong \mathsf{List}_{\text{flex}}$ *at all.* Existence of such an isomorphism would imply $\mathbf{GMon} \cong \mathbf{EM}(\mathsf{List}_{\text{flex}}) \cong \mathbf{EM}(\lceil\mathsf{List}\rceil) \cong \mathbf{EM}(\mathsf{List})$ over $\mathbf{GObj}_{\mathbb{E}}(\mathbf{Set})$, and would therefore contradict the fact that no rigidly graded monad has graded monoids as algebras, which we prove as the following theorem.

Theorem 4. *There is no rigidly* $\mathbb{N}^{\times}_{\leq}$*-graded monad* T *on* **Set** *such that* $\mathbf{GMon} \cong \mathbf{EM}(\mathsf{T})$ *over* $\mathbf{GObj}_{\mathbb{N}^{\times}_{\leq}}(\mathbf{Set})$*.*

Proof. By Theorem 2, to give an isomorphism $\mathbf{GMon} \cong \mathbf{EM}(\mathsf{T})$ over $\mathbf{GObj}_{\mathbb{N}^{\times}_{\leq}}(\mathbf{Set})$ is equivalent to giving an isomorphism $\mathbf{EM}(\mathsf{List}_{\text{flex}}) \cong \mathbf{EM}(\mathsf{T})$ over $\mathbf{GObj}_{\mathbb{N}^{\times}_{\leq}}(\mathbf{Set})$, so, by Corollary 1, it suffices to show that $R_{\mathsf{List}_{\text{flex}}} : \mathbf{EM}(\mathsf{List}_{\text{flex}}) \to \mathbf{EM}(\lfloor\mathsf{List}_{\text{flex}}\rfloor)$ is not an isomorphism. We can calculate that $Q_{\mathsf{List}} \cdot \mathbf{EM}(\alpha) = \mathbf{EM}(\psi) \cdot R_{\mathsf{List}_{\text{flex}}}$ where $\alpha : \lceil\mathsf{List}\rceil \to \mathsf{List}_{\text{flex}}$ is as above, and ψ is the isomorphism $\mathsf{List} \cong \lfloor\mathsf{List}_{\text{flex}}\rfloor$ from Example 9. Both Q_{List} and $\mathbf{EM}(\psi)$ are isomorphisms, but $\mathbf{EM}(\alpha)$ is not (by Lemma 1, since α is not an isomorphism). It follows that $R_{\mathsf{List}_{\text{flex}}}$ is not an isomorphism either.

Remark 4. One may ask whether it would make any difference to weaken existence of an isomorphism $\mathbf{GMon} \cong \mathbf{EM}(\mathsf{T})$ commuting strictly with the forgetful functors, to existence of an equivalence commuting up to natural isomorphism with the forgetful functors. It does not, because existence of the latter implies existence of the former. We do not attempt to determine whether there exists an equivalence $\mathbf{GMon} \simeq \mathbf{EM}(\mathsf{T})$ that does not commute with the forgetful functors. Such an equivalence would not enable us to make the carrier of a given T-algebra into a graded monoid, so is not useful for what we are trying to achieve.

Example 13. The rigidly $\mathbb{Z}^+_\leq$-graded monad Count has a flexibly graded extension $\lceil\mathsf{Count}\rceil$, defined by

$$\lceil\mathrm{Count}\rceil X e = \{t : \textstyle\prod_{i:\mathbb{N}} \coprod_{j:\mathbb{N}} X(e - \max\{0, j-i\}) \mid \\ \exists \rho \in \mathbb{N}.\, \forall k, j \in \mathbb{N}, x.\, t\,\rho = (j, x) \Rightarrow t(\rho + k) = (j+k, x)\}$$

and with similar unit and Kleisli extension to $\mathsf{Count}_{\mathrm{flex}}$.

We construct the isomorphism $Q_{\mathsf{Count}} : \mathbf{EM}(\lceil\mathsf{Count}\rceil) \cong \mathbf{EM}(\mathsf{Count})$. Given a $\lceil\mathsf{Count}\rceil$-algebra $(A, (-)^{\ddagger})$, the corresponding Count-algebra $(A, (-)^{\ddagger'})$ is defined by $f^{\ddagger'}_d\, t = f^{\ddagger}_d t$, where on the left we view t as an element of $\mathrm{Count}Xd$, and on the right we view t as an element of $\lceil\mathrm{Count}\rceil(J_{\mathbf{Set}}X)d$. In the other direction, we construct $(-)^{\ddagger}$ from $(-)^{\ddagger'}$. First note that the latter can be seen as an operator

$$\frac{h : Z \to Ae}{h^{\ddagger''} : \mathrm{Count}Z \Rightarrow A(- + e)}$$

Given $f : X \Rightarrow A(- + e)$ and $t \in \lceil\mathrm{Count}\rceil Xd$, let ρ be a witness to the side-condition on t in the definition of $\lceil\mathrm{Count}\rceil Xd$, and set $(j_i, x_i) = t\,i$ and $m_i = \max\{0, j_i - i\}$. (It does not matter which ρ is chosen.) Define $g : [0..\rho] \to A(d+e)$ by

$$g\,i = (f_{d-m_i})^{\ddagger''}_{m_i}(\lambda i'.\,(\max\{0, i' + (j_i - i)\}, x_i))$$

so $g^{\ddagger''}_0 : \mathrm{Count}[0..\rho]0 \to A(d+e)$, and then define $f^{\ddagger}_d t$ by

$$f^{\ddagger}_d t = g^{\ddagger''}_0(\lambda i.\,(i, \min\{i, \rho\}))$$

We show that graded arithmoids are not the algebras for any rigidly graded monad, using a similar argument to the argument for graded monoids above. There is a morphism $\beta : \lceil\mathsf{Count}\rceil \to \mathsf{Count}_{\mathrm{flex}}$ of flexibly graded monads, given by

$$\beta_{X,e}(\lambda i.\,(j_i, x_i)) = \lambda i.\,(j_i, X(e - \max\{0, j_i - i\} \leq e - (j_i - i))x_i)$$

This is not an isomorphism. To see this, define a graded set $Z : \mathbb{Z}_\leq \to \mathbf{Set}$ by $Zn = \{m \in \mathbb{Z} \mid m \leq n\}$. Then

$$(\lambda i.\,\mathrm{if}\ i = 0\ \mathrm{then}\ (0,0)\ \mathrm{else}\ (i-1, 1)) \in \mathrm{Count}_{\mathrm{flex}} Z0$$

is not in the image of $\beta_{Z,0}$. This implies the following theorem.

Theorem 5. *There is no rigidly $\mathbb{Z}^+_\leq$-graded monad* T *on* $\mathbf{Set}$ *such that* $\mathbf{GArith} \cong \mathbf{EM}(\mathsf{T})$ *over* $\mathbf{GObj}_{\mathbb{Z}^+_\leq}(\mathbf{Set})$.

Proof. By similar reasoning to the proof of Theorem 4, existence of such an isomorphism would imply that $\beta : \lceil\mathsf{Count}\rceil \to \mathsf{Count}_{\mathrm{flex}}$ is an isomorphism, which would be a contradiction.

6.1 Constructing Extensions

We turn to the problem of *constructing* the flexibly $\mathbb{E}$-graded monad $\lceil \mathsf{T} \rceil$ for a given rigidly $\mathbb{E}$-graded monad T on $\mathbb{C}$. It turns out that $\lceil \mathsf{T} \rceil$ exists exactly when certain (small) colimits exist in $\mathbf{EM}(\mathsf{T})$. We introduce the following class of (small) colimits in locally graded categories, which we use to construct $\lceil \mathsf{T} \rceil$.

Definition 20. *Let $\mathcal{D}$ be a locally $\mathbb{E}$-graded category, and let Y be an $\mathbb{E}$-graded object of $\underline{\mathcal{D}}$. The* internalization *of Y, if it exists, consists of an object $\mathrm{colim}^{\mathbb{E}}\, Y$ and natural family $(\lambda_d : Yd \mathrel{-d{\to}} \mathrm{colim}^{\mathbb{E}}\, Y)_{d \in \mathbb{E}}$ of morphisms in $\mathcal{D}$, universal in the sense that, for every $e \in |\mathbb{E}|$, $Z \in |\mathcal{D}|$, and natural family $(f_d : Yd \mathrel{-d \cdot e{\to}} Z)_{d \in \mathbb{E}}$, there is a unique $[f] : \mathrm{colim}^{\mathbb{E}}\, Y \mathrel{-e{\to}} Z$ such that $f_d = [f] \circ \lambda_d$ for all $d \in |\mathbb{E}|$.*

Here *naturality* of a family $(f_d : Yd \mathrel{-d \cdot e{\to}} Z)_{d \in \mathbb{E}}$ means $f_{d'} \circ Y\zeta = (\zeta \cdot e)^* f_d$ for all $\zeta \in \mathbb{E}(d, d')$. The universal property of $\mathrm{colim}^{\mathbb{E}}\, Y$ can be succinctly written as

$$\mathcal{D}(\mathrm{colim}^{\mathbb{E}}\, Y, Z)e \;\cong\; \int_{d \in \mathbb{E}} \mathcal{D}(Yd, Z)(d \cdot e) \qquad \text{naturally in } Z,\, e$$

where the naturality in Z is locally graded and the integral on the right is an end in **Set**; the elements of the right-hand side are the natural families $(f_d : Yd \mathrel{-d \cdot e{\to}} Z)_{d \in \mathbb{E}}$.

Example 14. Every graded object $X : \mathbb{E} \to \mathbb{C}$ of an ordinary category $\mathbb{C}$ induces a graded object $J_{\mathbb{C}}(X-)$ of $\underline{\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})}$. A natural family $(f_d : J_{\mathbb{C}}(Xd) \mathrel{-d \cdot e{\to}} Z)_{d \in \mathbb{E}}$ is a family of morphisms $f_{d,d'} : \mathbb{E}(1, d') \bullet Xd \to Z(d' \cdot d \cdot e)$ natural in d, d'; by the Yoneda lemma, these are in bijection with natural transformations $X \Rightarrow Z(- \cdot e)$, i.e. morphisms $X \mathrel{-e{\to}} Z$ in $\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$. Hence the internalization $\mathrm{colim}^{\mathbb{E}}(J_{\mathbb{C}}(X-))$ is just X equipped with the family λ corresponding to $\mathrm{id}_X : X \mathrel{-1{\to}} X$.

More generally, let $F : \mathbf{Free}_{\mathbb{E}}(\mathbb{C}) \to \mathcal{D}$ be a functor. Then, for each $X : \mathbb{E} \to \mathbb{C}$, we have a graded object $F(X-) : \mathbb{E} \to \underline{\mathcal{D}}$. If $\mathrm{colim}^{\mathbb{E}}(F(X-))$ exists for all X, then they form a functor $(X \mapsto \mathrm{colim}^{\mathbb{E}}(F(X-))) : \mathbf{GObj}_{\mathbb{E}}(\mathbb{C}) \to \mathcal{D}$. The latter is exactly the (pointwise) left Kan extension of F along $J_{\mathbb{C}}$ (in the enriched sense). We can therefore compute left Kan extensions along $J_{\mathbb{C}}$ as small colimits (even though $\mathbf{Free}_{\mathbb{E}}(\mathbb{C})$ might not be small). Example 14 above, where we take $F = J_{\mathbb{C}}$, shows that $\mathrm{Lan}_{J_{\mathbb{C}}} J_{\mathbb{C}}$ is the identity functor; in other words, that $J_{\mathbb{C}}$ is *dense.*

We can now construct $\lceil \mathsf{T} \rceil$ as follows. First construct the left Kan extension of the free T-algebra functor $F_{\mathsf{T}} : \mathbf{Free}_{\mathbb{E}}(\mathbb{C}) \to \mathbf{EM}(\mathsf{T})$ along $J_{\mathbb{C}}$, to obtain the left adjoint $\bar{F}_{\mathsf{T}} : \mathbf{GObj}_{\mathbb{E}}(\mathbb{C}) \to \mathbf{EM}(\mathsf{T})$ of the forgetful functor U_{T}. Then the composition $U_{\mathsf{T}} \cdot \bar{F}_{\mathsf{T}}$ forms a flexibly graded monad; this is $\lceil \mathsf{T} \rceil$. (Here we mean *left adjoint* in the usual enriched sense, in other words, the $\mathrm{Id}_{\mathbf{GObj}_{\mathbb{E}}(\mathbb{C})}$-relative left adjoint.)

Theorem 6. *A rigidly $\mathbb{E}$-graded monad T has a flexibly graded extension $\lceil \mathsf{T} \rceil$ if and only if $\mathrm{colim}^{\mathbb{E}}(F_{\mathsf{T}}(X-))$ exists in $\mathbf{EM}(\mathsf{T})$ for every $X : \mathbb{E} \to \underline{\mathbf{EM}(\mathsf{T})}$. When these exist, the functor*

$$\bar{F}_{\mathsf{T}} : X \mapsto \mathrm{colim}^{\mathbb{E}}(F_{\mathsf{T}}(X-)) : \mathbf{GObj}_{\mathbb{E}}(\mathbb{C}) \to \mathbf{EM}(\mathsf{T})$$

forms the left adjoint of U_{T}, and $\lceil\mathsf{T}\rceil$ is the flexibly graded monad induced by this adjunction.

Proof. The extension exists exactly when $U_{\mathsf{T}} : \mathbf{EM}(\mathsf{T}) \to \mathbf{GObj}_{\mathbb{E}}(\mathbb{C})$ is strictly monadic, and, by a general result about relative monads, this is the case exactly when U_{T} has a left adjoint. (If the left adjoint exists, the adjunction induces a flexibly graded monad $\lceil\mathsf{T}\rceil$, and functors $Q_{\mathsf{T}} : \mathbf{EM}(\lceil\mathsf{T}\rceil) \to \mathbf{EM}(\mathsf{T})$ and Q_{T}^{-1} : $\mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\lceil\mathsf{T}\rceil)$ can be constructed and shown to be inverses using the fact that Eilenberg-Moore resolutions are terminal.) Consider the following:

$$
\begin{array}{lll}
\multicolumn{2}{l}{\mathbf{EM}(\mathsf{T})(\bar{F}_{\mathsf{T}}X, \mathsf{A})e} & \\
\cong & \int_{d\in\mathbb{E}} \mathbf{EM}(\mathsf{T})(F_{\mathsf{T}}(Xd), \mathsf{A})(d\cdot e) & \text{universal property of } \mathrm{colim}^{\mathbb{E}} \\
\cong & \int_{d\in\mathbb{E}} \mathbf{GObj}_{\mathbb{E}}(\mathbb{C})(J_{\mathbb{C}}(Xd), U_{\mathsf{T}}\mathsf{A})(d\cdot e) & F_{\mathsf{T}} \text{ left } J_{\mathbb{C}}\text{-relative adjoint to } U_{\mathsf{T}} \\
\cong & \mathbf{GObj}_{\mathbb{E}}(\mathbb{C})(\mathrm{colim}^{\mathbb{E}}_{d}(J_{\mathbb{C}}(Xd)), U_{\mathsf{T}}\mathsf{A})e & \text{universal property of } \mathrm{colim}^{\mathbb{E}} \\
\cong & \mathbf{GObj}_{\mathbb{E}}(\mathbb{C})(X, U_{\mathsf{T}}\mathsf{A})e & \text{Example 14}
\end{array}
$$

The first isomorphism exists when $\bar{F}_{\mathsf{T}}X$ does, the others always exist. Hence the left adjoint must necessarily be $\bar{F}_{\mathsf{T}}$.

Remark 5. To justify our use of the word "colimit" for the internalization $\mathrm{colim}^{\mathbb{E}}\, Y$ of $Y : \mathbb{E} \to \underline{\mathcal{D}}$, we note that, if we view $\mathcal{D}$ as an $[\mathbb{E}, \mathbf{Set}]$-category (using Remark 1), then internalizations are a special case of *weighted* colimits in $\mathcal{D}$. To be more specific, let $\mathbb{E}^{\mathrm{rev}}$ be the monoidal category $\mathbb{E}$ but with the arguments of the tensor swapped. By the universal property of free locally graded categories, there are unique functors $W : \mathbf{Free}_{\mathbb{E}^{\mathrm{rev}}}(\mathbb{E}^{\mathrm{op}}) \to \mathbf{GObj}_{\mathbb{E}^{\mathrm{rev}}}(\mathbf{Set})$ and $Y^{\sharp} : \mathbf{Free}_{\mathbb{E}}(\mathbb{E}) \to \mathcal{D}$ such that $\underline{W} \cdot H_{\mathbb{E}^{\mathrm{op}}} = \mathrm{Hom}_{\mathbb{E}} : \mathbb{E}^{\mathrm{op}} \to [\mathbb{E}, \mathbf{Set}]$ and $\underline{Y^{\sharp}} \cdot H_{\mathbb{E}} = Y$. (Recall that $H_{\mathbb{C}} : \mathbb{C} \to \underline{\mathbf{Free}_{\mathbb{E}}(\mathbb{C})}$.) Then $\mathrm{colim}^{\mathbb{E}}\, Y$ is the colimit of $Y^{\sharp}$ weighted by W. This is a small colimit in $\mathcal{D}$, so it exists whenever $\mathcal{D}$ is cocomplete in the enriched sense. (Here the enriching category is not symmetric in general; for a definition of weighted colimit that does not assume symmetry, see [8].)

7 Related Work

Relative monads Relative monads were defined for ordinary categories by Altenkirch et al. [1], and generalized to $\mathbb{V}$-categories (for $\mathbb{V}$ symmetric monoidal) by Staton [23]. Our definitions of relative monad, algebra, and Kleisli (locally graded) category are generalizations of theirs. Our definition is not an instance of Lobbia's [14] generalization of relative monads to arbitrary 2-categories. Altenkirch et al. [1] studied the problem of extending a J-relative monad to a monad—as we do in Section 6. They defined a notion of well-behavedness for a functor J, which provides a sufficient condition for the extension to exist; when J is well-behaved, the extension $\lceil\mathsf{T}\rceil$ of T has as underlying functor $\lceil T\rceil$ the left Kan extension of T along J. We cannot use this result to construct flexibly graded extensions, because $J_{\mathbb{C}}$ is not well-behaved (in the appropriate locally graded sense). Hence we give an alternative construction of $\lceil \mathsf{T} \rceil$ (involving the Kan extension of F_{T} instead of T). In our case, the underlying functor of $\lceil \mathsf{T} \rceil$ is not $\mathrm{Lan}_{J_{\mathbb{C}}} T$ in general ($\lceil \mathsf{List} \rceil$ is a counterexample).

Graded monads Graded monads were introduced independently by Smirnov [22], by Melliès [16], and by Katsumata [9]. A formal theory for graded monads was first developed using actegories by Fujii et al. [4] (based on Street's [24] formal theory of *lax functors*). Presentations of graded monads have been studied by various authors [22,17,2,12], but these are all rigid, in that they present algebras of rigidly graded monads (so are not general enough to capture graded monoids or graded arithmoids).

Locally graded categories Locally graded categories were first introduced by Wood [26], who proved that they are enriched categories. We use the terminology of Levy [13]. They were also used in connection with grading by Melliès [16] and by Gaboardi et al. [5]. The latter in particular define the Kleisli locally graded category of a graded monad. The formulation of graded monads *within* locally graded category theory, which enables our development, is new here.

8 Conclusions

Graded monads cannot capture certain structures, such as graded monoids, as their algebras. This is the case even if their free algebras form instances of these structures. We show however that even when these structures are not captured exactly, we can often characterize the graded monad by a universal property, from which we can extract the structure of the free algebras. The proof of this involves the notion of flexibly graded monad. We introduce these primarily as a graded-monad-like tool for capturing these structures, though they may be useful in their own right as a generalization (modulo existence) of (rigidly) graded monads. We work within locally graded category theory, which provides a rich source of results for reasoning about grading.

As we state in the introduction, our primary motivation for this work is to develop a notion of presentation for graded monads that captures, for example, the operations of a graded monoid. This paper lays the groundwork for such a development, which we present in the sequel paper [10].

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References

1. Altenkirch, T., Chapman, J., Uustalu, T.: Monads need not be endofunctors. Log. Methods Comput. Sci **11**(1), 3:1–3:40 (2015). https://doi.org/10.2168/lmcs-11(1:3)2015

2. Dorsch, U., Milius, S., Schröder, L.: Graded monads and graded logics for the linear time - branching time spectrum. In: Fokkink, W., van Glabbeek, R. (eds.) Proc. of 30th Int. Conf. on Concurrency Theory, CONCUR 2019, Leibniz Int. Proc. in Informatics, vol. 140, pp. 36:1–36:16. Dagstuhl Publishing, Saarbrücken/Wadern (2019). https://doi.org/10.4230/lipics.concur.2019.36
3. Fritz, T., Perrone, P.: A probability monad as the colimit of spaces of finite samples. Theor. Appl. Categ. **34**(7), 170–220 (2019), http://www.tac.mta.ca/tac/volumes/34/7/34-07abs.html
4. Fujii, S., Katsumata, S., Melliès, P.A.: Towards a formal theory of graded monads. In: Jacobs, B., Löding, C. (eds.) FoSSaCS 2016, LNCS, vol. 9634, pp. 513–530. Springer, Cham (2016). https://doi.org/10.1007/978-3-662-49630-5_30
5. Gaboardi, M., Katsumata, S., Orchard, D., Sato, T.: Graded Hoare logic and its categorical semantics. In: Yoshida, N. (ed.) ESOP 2021, LNCS, vol. 12648, pp. 234–263. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-72019-3_9
6. Garner, R.: An embedding theorem for tangent categories. Adv. Math. **323**, 668–687 (2018). https://doi.org/10.1016/j.aim.2017.10.039
7. Goncharov, S.: Trace semantics via generic observations. In: Heckel, R., Milius, S. (eds.) CALCO 2013, LNCS, vol. 8089, pp. 158–174. Springer, Berlin, Heidelberg (2013). https://doi.org/10.1007/978-3-642-40206-7_13
8. Gordon, R., Power, A.: Algebraic structure for bicategory enriched categories. J. Pure Appl. Algebra **130**(2), 119–132 (1998). https://doi.org/10.1016/s0022-4049(97)00094-7
9. Katsumata, S.: Parametric effect monads and semantics of effect systems. In: Proc. of 41st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL '14, pp. 633–645. ACM Press, New York (2014). https://doi.org/10.1145/2535838.2535846
10. Katsumata, S., McDermott, D., Uustalu, T., Wu, N.: Flexible presentations of graded monads. Proc. ACM Program. Lang. **6**(ICFP), 123:1–123:29 (2022). https://doi.org/10.1145/3547654
11. Kelly, G.M.: Basic Concepts of Enriched Category Theory, London Math. Soc. Lecture Note Series, vol. 64. Cambridge University Press, Cambridge (1982), reprinted as: Reprints Theor. Appl. Categ. **10** (2005), http://www.tac.mta.ca/tac/reprints/articles/10/tr10abs.html
12. Kura, S.: Graded algebraic theories. In: Goubault-Larrecq, J., König, B. (eds.) FoSSaCS 2020, LNCS, vol. 12077, pp. 401–421. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-45231-5_21
13. Levy, P.B.: Locally graded categories. Slides (2019), https://www.cs.bham.ac.uk/~pbl/papers/locgrade.pdf
14. Lobbia, G.: Distributive laws for relative monads. arXiv preprint arXiv:2007.12982 [math.CT] (2020), https://arxiv.org/abs/2007.12982
15. Marmolejo, F., Wood, R.J.: Monads as extension systems: No iteration is necessary. Theor. Appl. Categ. **24**(4), 84–113 (2010), http://www.tac.mta.ca/tac/volumes/24/4/24-04abs.html
16. Melliès, P.A.: Parametric monads and enriched adjunctions. Manuscript (2012), https://www.irif.fr/~mellies/tensorial-logic/8-parametric-monads-and-enriched-adjunctions.pdf
17. Milius, S., Pattinson, D., Schröder, L.: Generic trace semantics and graded monads. In: Moss, L.S., Sobociński, P. (eds.) Proc. of 6th Conf. on Algebra and Coalgebra in Computer Science, CALCO 2015, Leibniz Int. Proceedings in Informatics, vol. 35, pp. 253–269. Dagstuhl Publishing, Saarbrücken/Wadern (2015). https://doi.org/10.4230/lipics.calco.2015.253

18. Moggi, E.: Computational lambda-calculus and monads. In: Proc. of 4th Ann. Symp. on Logic in Computer Science, LICS '89, pp. 14–23. IEEE Press, Los Alamitos, CA (1989). https://doi.org/10.1109/lics.1989.39155
19. Moggi, E.: Notions of computation and monads. Inf. Comput. **93**(1), 55–92 (1991). https://doi.org/10.1016/0890-5401(91)90052-4
20. Mycroft, A., Orchard, D., Petricek, T.: Effect systems revisited—control-flow algebra and semantics. In: Probst, C.W., Hankin, C., Hansen, R.R. (eds.) Nielsons' Festschrift, LNCS, vol. 9560, pp. 1–32. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-27810-0_1
21. Plotkin, G., Power, J.: Algebraic operations and generic effects. Appl. Categ. Struct. **11**, 69–94 (2003). https://doi.org/10.1023/a:1023064908962
22. Smirnov, A.: Graded monads and rings of polynomials. J. Math. Sci. **151**(3), 3032–3051 (2008). https://doi.org/10.1007/s10958-008-9013-7
23. Staton, S.: An algebraic presentation of predicate logic. In: Pfenning, F. (ed.) FoSSaCS 2013, LNCS, vol. 7794, pp. 401–417. Springer, Berlin, Heidelberg (2013). https://doi.org/10.1007/978-3-642-37075-5_26
24. Street, R.: Two constructions on lax functors. Cah. Topol. Géom. Diff. Catég. **13**(3), 217–264 (1972), http://www.numdam.org/item/CTGDC_1972__13_3_217_0
25. Ulmer, F.: Properties of dense and relative adjoint functors. J. Algebra **8**(1), 77–95 (1968). https://doi.org/10.1016/0021-8693(68)90036-7
26. Wood, R.J.: Indicial Methods for Relative Categories. Ph.D. thesis, Dalhousie University (1976), http://hdl.handle.net/10222/55465