Canonical Gradings of Monads

Flavien Breuvart
LIPN, Université Sorbonne Paris Nord, France
flavien.breuvart@lipn.univ-paris13.fr

Dylan McDermott
Reykjavik University, Iceland
dylanm@ru.is

Tarmo Uustalu
Reykjavik University, Iceland
Tallinn University of Technology, Estonia
tarmo@ru.is

We define a notion of grading of a monoid T in a monoidal category $\mathscr{C}$, relative to a class of morphisms $\mathscr{M}$ (which provide a notion of $\mathscr{M}$-subobject). We show that, under reasonable conditions (including that $\mathscr{M}$ forms a factorization system), there is a canonical grading of T. Our application is to graded monads and models of computational effects. We demonstrate our results by characterizing the canonical gradings of a number of monads, for which $\mathscr{C}$ is endofunctors with composition. We also show that we can obtain canonical grades for algebraic operations.

1 Introduction

This paper is motivated by quantitative modelling of computational effects from mathematical programming semantics. It is standard in this domain to model notions of computational effect, such as nondeterminism or manipulation of external state, by (strong) monads [11]. In many applications, however, it is useful to be able to work with quantified effects, e.g., how many outcomes a computation may have, or to what degree it may read or overwrite the state. This is relevant, for example, for program optimizations or analyses to assure that a program can run within allocated resources. Quantification of effectfulness is an old idea and goes back to type-and-effect systems [8]. Mathematically, notions of quantified effect can be modelled by graded (strong) monads [13, 10, 4].

It is natural to ask if there are systematic ways for refining a non-quantitative model of some effect into a quantitative version, i.e., for producing a graded monad from a monad. In this paper, we answer this question in the affirmative. We show how a monad on a category can be graded with any class of subfunctors (intuitively, predicates on computations) satisfying reasonable conditions, including that it forms a factorization system on some monoidal subcategory of the endofunctor category, moreover, this grading is canonical, namely universal in a certain 2-categorical sense. We also show that algebraic operations of the given monad give rise to canonical algebraic operations of the graded monad that are flexibly graded in the sense of [5]. We show this by abstracting from monads to monoids in a (skew) monoidal category with a factorization system.

The structure of the paper is this. In Section 2, we introduce the idea of grading by subobjects for general objects and instantiate this for grading of functors. We then proceed to gradings of monoids and monads in Section 3. In Section 4, we explore the specific interesting case of grading monads canonically by subsets of their sets of shapes. In Section 5, we explain the emergence of canonical flexibly graded algebraic operations for canonical gradings of monads. One longer proof is in Appendix A.

We introduce the necessary concepts regarding the classical topics of monads, monoidal categories and factorization systems. For additional background on the more specific concepts of graded monad and skew monoidal category, which we also introduce, we refer to [4, 2] and [14, 7] as entry points.

Preliminary Report. Final version to appear in:
ACT 2022

2 Grading objects and functors

As a first step towards gradings of monoids, we introduce the notion of a grading of an object of a category $\mathscr{C}$ with respect to a class of morphisms $\mathscr{M}$ in $\mathscr{C}$. We show that every object X has a canonical such grading. The category $\mathscr{G}$ of grades is in this case $\mathscr{M}/X$, which has as objects pairs (S,s) of an object S and an $\mathscr{M}$-morphism $s : S \rightarrowtail X$; morphisms $f : (S,s) \to (S',s')$ are $\mathscr{C}$-morphisms $f : S \to S'$ such that $s = s' \circ f$. The case we care most about is when $\mathscr{C}$ is a category of endofunctors (so that in the next section, where we extend these results to gradings of monoids, the monoids are exactly monads).

Definition 2.1. A $\mathscr{G}$-graded object of a category $\mathscr{C}$ is a functor $G : \mathscr{G} \to \mathscr{C}$.

Definition 2.2. Let $\mathscr{M}$ be a class of morphisms of a category $\mathscr{C}$. An *$\mathscr{M}$-grading* $(\mathscr{G}, G, g)$ of an object X of $\mathscr{C}$ consists of a category $\mathscr{G}$, a functor $G : \mathscr{G} \to \mathscr{C}$ (= a $\mathscr{G}$-graded object of $\mathscr{C}$), and a natural transformation typed $g_d : Gd \rightarrowtail X$ whose components are all in $\mathscr{M}$. A *morphism* $(F,f) : (\mathscr{G},G,g) \to (\mathscr{G}',G',g')$ between such gradings is a functor $F : \mathscr{G} \to \mathscr{G}'$ equipped with a natural isomorphism $f : G' \cdot F \cong G$, such that

$$\begin{tikzcd} \mathscr{G} \arrow[r] \arrow[d, "F"'] \arrow[dr, "G" description] \arrow[dr, phantom, shift left=4, "\overset{g}{\Rightarrow}"] \arrow[dr, phantom, shift right=4, "\overset{f}{\Rightarrow}"] & \mathbf{1} \arrow[d, "X"] \\ \mathscr{G}' \arrow[r, "G'"'] & \mathscr{C} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathscr{G} \arrow[r] \arrow[d, "F"'] & \mathbf{1} \arrow[d, "X"] \\ \mathscr{G}' \arrow[ur] \arrow[r, "G'"'] \arrow[r, phantom, shift left=3, "\overset{g'}{\Rightarrow}"] & \mathscr{C} \end{tikzcd}$$

A *2-cell* $\beta : (F,f) \Rightarrow (F',f')$ between such morphisms is a natural transformation $\beta : F \Rightarrow F'$ such that $f' \circ (G' \cdot \beta) = f$. These form a 2-category $\mathbf{Grade}_{\mathscr{M}} X$.

We define a 2-category of gradings so that we can prove a universal property (the following theorem) that characterizes the canonical grading up to equivalence. There is no reason to distinguish between isomorphic grades, and so we often work with gradings that are equivalent to the canonical one. (Working up to equivalence has the added benefit that, since $\mathscr{M}/X$ is often equivalent to a small category, we often have a canonical grading with a small set of grades.)

Theorem 2.3. *Let $\mathscr{M}$ be a class of a morphisms of a category $\mathscr{C}$, and let X be an object of $\mathscr{C}$. The data $(\mathscr{M}/X, X_{\mathscr{M}}, \mathrm{snd})$, where $X_{\mathscr{M}}(S,s) = S$ and $\mathrm{snd}(S,s) = s$, make a grading of X. This grading is canonical in the sense that it is the pseudoterminal object of $\mathbf{Grade}_{\mathscr{M}} X$. Explicitly, for every other $\mathscr{M}$-grading $(\mathscr{G}, G, g)$ of X, there is, up to isomorphism, a unique morphism $(F, f) : (\mathscr{G}, G, g) \to (\mathscr{M}/X, X_{\mathscr{M}}, \mathrm{snd})$ of $\mathscr{M}$-gradings.*

Proof. For existence, define $(F,f) : (\mathscr{G},G,g) \to (\mathscr{M}/X, X_{\mathscr{M}}, \mathrm{snd})$ by

$$Fd = (Gd, g_d) \quad Fh = Gh \qquad f_d = \mathrm{id}_{Gd}$$

For uniqueness, given (F',f'), we have 2-cells $\beta : (F',f') \Rightarrow (F,f)$ and $\beta^{-1} : (F,f) \Rightarrow (F',f')$ given by $\beta_d = f'_d$ and $\beta_d^{-1} = f_d'^{-1}$. These are clearly inverse to each other, so β is the required isomorphism $(F',f') \cong (F,f)$. □

Remark 2.4. In this paper, we discuss the problem of constructing canonical gradings, but one can also consider the dual problem of constructing canonical *degradings*. For graded monads, this problem is discussed in [1, 9]. In the setting of this section, the initial degrading of a functor $G : \mathscr{G} \to \mathscr{C}$ would be the colimit $\mathrm{colim}\, G$, together with the morphisms $\mathrm{in}_e : Ge \to \mathrm{colim}\, G$ (when the colimit exists). The data $(\mathscr{G}, G, \mathrm{in}_{)}$ is then an $\mathscr{M}$-grading of $\mathrm{colim}\, G$ whenever in_e is in $\mathscr{M}$ for all e (which is the case for our examples). This grading will typically not be the canonical grading of $\mathrm{colim}\, G$ however. (For graded monads the situation is more complex: one does not take an ordinary colimit, but instead a colimit a 2-category of monoidal categories, as discussed in [1].)

2.1 Canonical gradings of endofunctors on Set

We give several examples for the case where $\mathscr{C} = [\mathbf{Set},\mathbf{Set}]$ and $\mathscr{M}$ is natural transformations whose components are injective functions. In this case, every $\mathscr{M}$-subobject of an endofunctor $F : \mathbf{Set} \to \mathbf{Set}$ is isomorphic in $\mathscr{M}/F$ to a unique $\mathscr{M}$-subobject $s : S \rightarrowtail F$ in which each injection s_X is an inclusion. Below we characterize $\mathscr{M}/F$ up to equivalence for various endofunctors F, using the fact that we need only consider the case where s is a family of inclusions.

Example 2.5. The category $\mathscr{M}/\mathrm{Id}$ is equivalently the poset $\{\bot \leq \top\}$, with $\bot$ corresponding to $SX = \emptyset$ for all X, and $\top$ corresponding to the $\mathscr{M}$-subobject given by $SX = X$ for all X.

Example 2.6. Consider the endofunctor $M \times F-$, where M is a set and F is an endofunctor on **Set**. The category $\mathscr{M}/(M \times F-)$, is equivalent to the category $(\mathscr{M}/F)^M$, in which objects are M-indexed families $(\Sigma_z \in \mathscr{M}/F)_{z \in M}$ of $\mathscr{M}$-subobjects of F. This is the case because, from every $S \rightarrowtail M \times F-$ in which each component is an inclusion, we can construct such a family $\mathbf{\Sigma}[S]$, and this construction forms a bijection with inverse $\mathbf{S}[-]$.

$$\mathbf{\Sigma}[S]_z X = \{x \in FX \mid (z,x) \in SX\} \qquad \mathbf{S}[\Sigma]X = \{(z,x) \in M \times FX \mid x \in \Sigma_z X\}$$

In the special case $F = \mathrm{Id}$, we have $\mathscr{M}/(M \times (-)) \simeq (\mathscr{M}/\mathrm{Id})^M \simeq \{\bot \leq \top\}^M \simeq (\mathscr{P}M, \subseteq)$, so the $\mathscr{M}$-subobjects of $M \times (-)$ are equivalently the subsets of M, ordered by inclusion.

Example 2.7. Consider the endofunctor $V \Rightarrow (-)$ (the underlying functor of the reader monad on **Set**), where V is a fixed set, and let $\mathscr{M}$ be the class of componentwise injective natural transformations. The $\mathscr{M}$-subobjects of $V \Rightarrow (-)$, and hence the objects of the canonical $\mathscr{M}$-grading of $V \Rightarrow (-)$, are equivalently upwards-closed sets of equivalence relations on V.

To explain this in more detail, let Equiv_V be the set of equivalence relations R on V, considered as subsets $R \subseteq V \times V$. A function $f : V \to X$ *respects* $R \in \mathrm{Equiv}_V$ when $v R v'$ implies $fv = fv'$ for all $v, v' \in V$, equivalently, when f factors through the quotient $[-]_R : V \to V/R$. A set $\Sigma \subseteq \mathrm{Equiv}_V$ of equivalence relations is *upwards-closed* when, $R \in \Sigma$ implies $R' \in \Sigma$ for all $R, R' \in \mathrm{Equiv}_V$ with $R \subseteq R'$. Every such Σ induces a subfunctor $\mathbf{S}[\Sigma] \rightarrowtail V \Rightarrow (-)$, defined by

$$\mathbf{S}[\Sigma]X = \{f : V \to X \mid f \text{ respects some } R \in \Sigma\}$$

To go in the other direction, consider a subfunctor $S \rightarrowtail V \Rightarrow (-)$ in which every component of the $\mathscr{M}$-morphism is an inclusion. We obtain an upwards-closed $\mathbf{\Sigma}[S] \subseteq \mathrm{Equiv}_V$:

$$\mathbf{\Sigma}[S] = \{R \in \mathrm{Equiv}_V \mid [-]_R \in S(V/R)\}$$

This is upwards-closed because if $R \subseteq R'$ then $[-]_{R'} : V \to V/R'$ factors through $[-]_R : V \to V/R$, and to form a functor, S must be closed under postcomposition. These two constructions are in bijection, with $\mathbf{S}[\mathbf{\Sigma}[S]] = S$ and $\mathbf{\Sigma}[\mathbf{S}[\Sigma]] = \Sigma$. It follows that $\mathscr{M}/(V \Rightarrow (-))$ is equivalent to the poset of upwards-closed sets $\Sigma \subseteq \mathrm{Equiv}_V$, ordered by inclusion, and hence that this poset forms the canonical $\mathscr{M}$-grading of $V \Rightarrow (-)$.

Example 2.8. Consider the endofunctor $V \Rightarrow V \times (-)$ (the underlying functor of the state monad), where V is a set. Since $V \Rightarrow V \times (-) \cong (V \Rightarrow V) \times (V \Rightarrow (-))$, we can combine Examples 2.6 and 2.7 to characterize the $\mathscr{M}$-subobjects of $V \Rightarrow V \times (-)$. Every such subobject equivalently consists of an upwards-closed set $\Sigma_p \subseteq \mathrm{Equiv}_V$ for each function $p : V \to V$. These can also be seen as subsets $\Sigma \subseteq (V \Rightarrow V) \times \mathrm{Equiv}_V$ such that $\{R \mid (p,R) \in \Sigma\}$ is upwards-closed for each $p : V \to V$. Given such a Σ, the corresponding $\mathscr{M}$-subobject $\mathbf{S}[\Sigma] \rightarrowtail (V \Rightarrow V \times (-))$ is

$$\mathbf{S}[\Sigma]X = \{f : V \to V \times X \mid \exists (p,R) \in \Sigma.\ \pi_1 \circ f = p \wedge \pi_2 \circ f \text{ respects } R\}$$

3 Grading monoids and monads

We proceed to grading monoids. This works more robustly in skew monoidal categories [14] than monoidal categories.

Definition 3.1. A *(left-)skew monoidal* category is a category $\mathscr{C}$ with a distinguished object I, a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and three natural transformations λ, ρ, α typed

$$\lambda_X : \mathsf{I} \otimes X \to X \qquad \rho_X : X \to X \otimes \mathsf{I} \qquad \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

satisfying the equations

$$\text{(m1)} \quad \begin{tikzcd} & \mathsf{I} \otimes \mathsf{I} \arrow[dr, "\lambda_\mathsf{I}"] & \\ \mathsf{I} \arrow[ur, "\rho_\mathsf{I}"] \arrow[rr, equal] & & \mathsf{I} \end{tikzcd} \qquad \text{(m2)} \quad \begin{tikzcd} (X \otimes \mathsf{I}) \otimes Y \arrow[r, "\alpha_{X,\mathsf{I},Y}"] & X \otimes (\mathsf{I} \otimes Y) \arrow[d, "X \otimes \lambda_Y"] \\ X \otimes Y \arrow[u, "\rho_X \otimes Y"] \arrow[r, equal] & X \otimes Y \end{tikzcd}$$

$$\text{(m3)} \quad \begin{tikzcd} (\mathsf{I} \otimes X) \otimes Y \arrow[rr, "\alpha_{\mathsf{I},X,Y}"] \arrow[dr, "\lambda_X \otimes Y"'] & & \mathsf{I} \otimes (X \otimes Y) \arrow[dl, "\lambda_{X \otimes Y}"] \\ & X \otimes Y & \end{tikzcd} \qquad \text{(m4)} \quad \begin{tikzcd} (X \otimes Y) \otimes \mathsf{I} \arrow[rr, "\alpha_{X,Y,\mathsf{I}}"] & & X \otimes (Y \otimes \mathsf{I}) \\ & X \otimes Y \arrow[ul, "\rho_{X \otimes Y}"] \arrow[ur, "X \otimes \rho_Y"'] & \end{tikzcd}$$

$$\text{(m5)} \quad \begin{tikzcd} (X \otimes (Y \otimes Z)) \otimes W \arrow[rr, "\alpha_{X,Y \otimes Z,W}"] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[d, "X \otimes \alpha_{Y,Z,W}"] \\ ((X \otimes Y) \otimes Z) \otimes W \arrow[u, "\alpha_{X,Y,Z} \otimes W"] \arrow[r, "\alpha_{X \otimes Y,Z,W}"] & (X \otimes Y) \otimes (Z \otimes W) \arrow[r, "\alpha_{X,Y,Z \otimes W}"] & X \otimes (Y \otimes (Z \otimes W)) \end{tikzcd}$$

$(\mathscr{C}, \mathsf{I}, \otimes)$ is partially normal if one or several of λ, ρ or α is a natural isomorphism. In particular, it is *left-normal* if λ is an isomorphism. A monoidal category is a fully normal skew monoidal category.

A *right-skew monoidal category* is given by $(\mathscr{C}, \mathsf{I}, \otimes, \lambda, \rho, \alpha)$ such that the data $(\mathscr{C}, \mathsf{I}, \otimes^{\mathrm{rev}}, \rho, \lambda, \alpha)$, where $X \otimes^{\mathrm{rev}} Y = Y \otimes X$, form a left-skew monoidal category.

Definition 3.2. A *monoid* in a skew monoidal category $(\mathscr{C}, \mathsf{I}, \otimes)$ is an object T of $\mathscr{C}$ with natural transformations

$$\eta : \mathsf{I} \to T \qquad \mu : T \otimes T \to T$$

satisfying the equations

$$\begin{tikzcd} T \arrow[d, "\rho_T"'] \arrow[ddrr, equal] & \mathsf{I} \otimes T \arrow[l, "\lambda_T"'] \arrow[r, "\eta \otimes T"] & T \otimes T \arrow[dd, "\mu"] \\ T \otimes \mathsf{I} \arrow[d, "T \otimes \eta"'] & & \\ T \otimes T \arrow[rr, "\mu"] & & T \end{tikzcd} \qquad \begin{tikzcd} (T \otimes T) \otimes T \arrow[r, "\mu \otimes T"] \arrow[d, "\alpha_{T,T,T}"'] & T \otimes T \arrow[dd, "\mu"] \\ T \otimes (T \otimes T) \arrow[d, "T \otimes \mu"'] & \\ T \otimes T \arrow[r, "\mu"] & T \end{tikzcd}$$

The concept of lax monoidal functor between skew monoidal categories is defined as for monoidal categories; the same applies to the concept of monoidal transformations between lax monoidal functors.

Definition 3.3. Given a (skew) monoidal category $\mathbb{G}$, a $\mathbb{G}$-*graded monoid* in a (skew) monoidal category $\mathbb{C} = (\mathscr{C}, \mathsf{I}, \otimes)$ is the same as a lax monoidal functor $\mathsf{G} : \mathbb{G} \to \mathbb{C}$. Explicitly, it is a functor $G : \mathscr{G} \to \mathscr{C}$ with a morphism $\eta : \mathsf{I} \to G\mathsf{I}$ and a natural transformation typed $\mu_{d,d'} : Gd \otimes Gd' \to G(d \odot d')$ subject to equations similar to those of a monoid.

Definition 3.4. Let $\mathsf{T} = (T, \eta, \mu)$ be a monoid in a (skew) monoidal category $\mathbb{C} = (\mathscr{C}, \mathsf{I}, \otimes)$, and let $\mathscr{M}$ be a class of morphisms of $\mathscr{C}$. An *$\mathscr{M}$-grading* $(\mathbb{G}, \mathsf{G}, g)$ of the monoid T consists of a skew monoidal category $\mathbb{G}$, a lax monoidal functor $\mathsf{G} : \mathbb{G} \to \mathbb{C}$ (= a $\mathbb{G}$-graded monoid in $\mathbb{C}$), and a monoidal transformation $g_d : Gd \Rightarrow T$, whose components are all in $\mathscr{M}$. A *morphism* $(\mathsf{F}, f) : (\mathbb{G}, \mathsf{G}, g) \to (\mathbb{G}', \mathsf{G}', g')$ between such gradings is a lax monoidal functor $\mathsf{F} : \mathbb{G} \to \mathbb{G}'$ equipped with a monoidal natural isomorphism $f : \mathsf{G}' \cdot \mathsf{F} \cong \mathsf{G}$, such that $g \circ f = g'$. A *2-cell* $\alpha : (\mathsf{F}, f) \Rightarrow (\mathsf{F}', f')$ is a monoidal natural isomorphism $\alpha : \mathsf{F} \Rightarrow \mathsf{F}'$ such that $f' \circ (\mathsf{G}' \cdot \alpha) = f$. We write $\mathbf{Grade}_{\mathscr{M}}\mathsf{T}$ for this 2-category.

Example 3.5. The situation we are mainly interested in is when $\mathscr{C} = [\mathscr{D}, \mathscr{D}]$ is the category of endofunctors on some $\mathscr{D}$, with the identity for I and functor composition for $\otimes$. In this case, monoids in $\mathbb{C}$ are exactly monads on $\mathscr{D}$, and a lax monoidal functor $\mathsf{G} : \mathbb{G} \to \mathbb{C}$ (a $\mathbb{G}$-graded monoid in $\mathbb{C}$) is a $\mathbb{G}$-graded monad on $\mathscr{D}$, in the sense of [13, 10, 4]. Explicitly, the unit and multiplication of G have the form

$$\eta_X : X \to GIX \qquad \mu_{e,e',X} : Ge(Ge'X) \to G(e \odot e')X$$

For a concrete example, let V be a set (of states), and let T be the state monad over V:

$$TX = V \Rightarrow V \times X \qquad \eta_X\, x\, v = (v, x) \qquad \mu_X\, f\, v = g\, v' \;\text{ where } (v', g) = f\, v$$

We give a $\mathscr{M}$-grading $(\mathbb{G}, \mathsf{G}, g)$ of T, where $\mathscr{M}$ is componentwise injective natural transformations. Let $\mathscr{G}$ be the poset of subsets of $\{\mathsf{get}, \mathsf{put}\}$ ordered by inclusion, which forms a strict monoidal category with $\emptyset$ for J and $e \cup e'$ for $e \boxdot e'$. We then define G by

$$\begin{aligned} Ge = \{ f : V \to V \times X \\ &\mid \mathsf{get} \notin e \Rightarrow (\pi_1 \circ f \text{ is a constant function or } \mathrm{id}_V \;\wedge\; \pi_2 \circ f \text{ is a constant function}) \\ &\;\; \wedge \mathsf{put} \notin e \Rightarrow \pi_1 \circ f \text{ is } \mathrm{id}_V \} \end{aligned}$$

with unit and multiplication defined as for T. This forms an $\mathscr{M}$-grading with the inclusions for g. (This grading is suitable for interpreting a Gifford-style effect system [8] for global state.)

If T is a monoid in $\mathbb{C}$, then the slice category $\mathscr{C}/T$ forms a skew monoidal category, with

$$\mathsf{I} \xrightarrow{\eta} T \qquad S \otimes S' \xrightarrow{s \otimes s'} T \otimes T \xrightarrow{\mu} T$$

for the unit and tensor of (S, s) and (S', s') (see for example Kelly [6]). In general this (skew) monoidal structure will not restrict to $\mathscr{M}/T$; the morphism μ is not in $\mathscr{M}$ for many of our examples. However, we can make $\mathscr{M}/T$ into a skew monoidal category by adapting the (skew) monoidal structure on $\mathscr{C}/T$. The idea is to just factorize the morphisms we use in the monoidal structure of $\mathscr{C}/T$ to obtain morphisms in $\mathscr{M}$. Hence we ask that $\mathscr{M}$ forms an orthogonal factorization system in the usual sense.

Definition 3.6. An (orthogonal) *factorization system* on a category $\mathscr{C}$ is a pair $(\mathscr{E}, \mathscr{M})$ of classes of morphisms of $\mathscr{C}$, such that

- both $\mathscr{E}$ and $\mathscr{M}$ contain all isomorphisms, and are closed under composition;
- $\mathscr{E}$-morphisms are *orthogonal* to $\mathscr{M}$-morphisms: for every commuting square in $\mathscr{C}$ as on the left below, with $e \in \mathscr{E}$ and $m \in \mathscr{M}$, there is a unique d making the diagram on the right below commute.

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, tail, "m"'] & Y' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "d"'] \\ X' \arrow[r, tail, "m"'] & Y' \end{tikzcd}$$

- every morphism $f : X \to Y$ in $\mathscr{C}$ has an $(\mathscr{E},\mathscr{M})$-factorization: there exist an object S, an $\mathscr{E}$-morphism $e : X \twoheadrightarrow S$, and an $\mathscr{M}$-morphism $m : S \rightarrowtail Y$ such that the diagram below commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & Y \\ & S \arrow[ur, tail, "m"'] & \end{tikzcd}$$

Example 3.7. Writing **Mor** for the class of all morphisms and **Iso** for the class of isomorphisms, every category has $(\mathbf{Mor},\mathbf{Iso})$ and $(\mathbf{Iso},\mathbf{Mor})$ as factorization systems. On **Set**, the classes of surjective and of injective functions form a factorization system $(\mathbf{Epi},\mathbf{Mono})$. To factorize a function $f : X \to Y$ in this case, we let $S = \{f\,x \mid x \in X\}$ be the image of f, and define $X \xrightarrow{e} S \xrightarrow{m} Y$ by $e\,x = f\,x$ and $m\,y = y$. On the category **Poset** of partially ordered sets and monotone functions, we have a factorization system $(\mathbf{Epi},\mathbf{Full})$, where **Epi** is the class of surjective monotone functions and **Full** is the class of full functions, i.e. monotone functions $m : S \to Y$ such that $m\,x \le m\,y$ implies $x \le y$. Factorizations are given as in **Set**, with the order on S inherited from the order on Y.

We are primarily interested in canonically grading monads, which are monoids in the endofunctor category $\mathscr{C} = [\mathscr{D},\mathscr{D}]$, with functor composition as the tensor. We therefore want a factorization system on $[\mathscr{D},\mathscr{D}]$. We give the most standard option for this as the following example, but there are others we are interested in (see Lemma 4.2 below). In models of computational effects we in fact usually want a *strong* monad. If $\mathscr{D}$ is monoidal, then a *strong endofunctor* on $\mathscr{D}$ is a functor $F : \mathscr{D} \to \mathscr{D}$ equipped with a *strength*, i.e. a natural transformation $\mathrm{str}_{\Gamma,X} : \Gamma \otimes_{\mathscr{D}} FX \to F(\Gamma \otimes_{\mathscr{D}} X)$ satisfying two laws for compatibility with the left unitor and associator of $\mathscr{D}$. These form a monoidal category $[\mathscr{D},\mathscr{D}]_s$, in which morphisms are strength-preserving natural transformations and the tensor is composition. Strong monads are monoids in $[\mathscr{D},\mathscr{D}]_s$. Below we consider non-strong monads for simplicity, but we can also apply our results to strong monads using a factorization system on $[\mathscr{D},\mathscr{D}]_s$.

Example 3.8. If $(\mathscr{E},\mathscr{M})$ is a factorization system on a category $\mathscr{D}$, then the endofunctor category $[\mathscr{D},\mathscr{D}]$ has a factorization system (componentwise-$\mathscr{E}$,componentwise-$\mathscr{M}$). Factorizations $F \twoheadrightarrow S \rightarrowtail G$ of natural transformations are componentwise. If $\mathscr{D}$ is monoidal and $\mathscr{E}$ is closed under $\Gamma \otimes_{\mathscr{D}} (-)$ for all Γ then (componentwise-$\mathscr{E}$,componentwise-$\mathscr{M}$) is a factorization system on $[\mathscr{D},\mathscr{D}]_s$. Morphisms are again factorized componentwise; for the construction of the strength for S see [3, Section 2.2].

Forming a factorization system $(\mathscr{E},\mathscr{M})$ is merely a property of a class $\mathscr{M}$ of morphisms, since $\mathscr{E}$ is necessarily the class of all morphisms e that are orthogonal to all $\mathscr{M}$-morphisms. Factorizations are unique up to canonical isomorphism.

If $\mathscr{M}$ forms a factorization system $(\mathscr{E},\mathscr{M})$, then for a given monoid T we construct a unit J and a tensor $\boxdot$ for the category $\mathscr{M}/T$ by factorizing morphisms as follows:

$$\begin{tikzcd} \mathsf{I} \arrow[rr, "\eta"] \arrow[dr, two heads, "q"'] & & T \\ & \mathsf{J} \arrow[ur, tail] & \end{tikzcd} \qquad \begin{tikzcd} S \otimes S' \arrow[r, "s\otimes s'"] \arrow[dr, two heads, "q_{S,S'}"'] & T \otimes T \arrow[r, "\mu"] & T \\ & S \boxdot S' \arrow[ur, tail] & \end{tikzcd}$$

To construct the required structural morphisms, we make the additional assumption that $\mathscr{E}$ is closed under $(-) \otimes S$ for every $S \rightarrowtail T$. Under this assumption, the following squares, which all commute, have a unique diagonal, and these diagonals are the required structural morphisms.

$$
\begin{tikzcd}
S_1 \otimes S'_1 \arrow[r, two heads, "q_{S_1,S'_1}"] \arrow[d, "f\otimes f'"'] & S_1 \boxdot S'_1 \arrow[dd, tail] \arrow[ddl, dashed, "f\boxdot f'"'] \\
S_2 \otimes S'_2 \arrow[d, two heads, "q_{S_2,S'_2}"'] & \\
S_2 \boxdot S'_2 \arrow[r, tail] & T
\end{tikzcd}
\quad \text{where} \quad
\begin{tikzcd}
S_1 \arrow[r, "f"] \arrow[dr, tail] & S_2 \arrow[d, tail] \\
& T
\end{tikzcd}
\quad
\begin{tikzcd}
S'_1 \arrow[r, "f'"] \arrow[dr, tail] & S'_2 \arrow[d, tail] \\
& T
\end{tikzcd}
\quad \text{are morphisms in } \mathscr{M}/T
$$

$$
\begin{tikzcd}
\mathsf{I}\otimes S \arrow[r, two heads, "q\otimes S"] \arrow[d, "\lambda_S"'] & \mathsf{J}\otimes S \arrow[r, two heads, "q_{\mathsf{J},S}"] & \mathsf{J}\boxdot S \arrow[d, tail] \arrow[dll, dashed, "\ell_S"'] \\
S \arrow[rr, tail] & & T
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, equal] \arrow[d, "\rho_S"'] & S \arrow[dddd, tail] \arrow[dddl, dashed, "r_S"] \\
S\otimes\mathsf{I} \arrow[d, "S\otimes q"'] & \\
S\otimes\mathsf{J} \arrow[d, two heads, "q_{S,\mathsf{J}}"'] & \\
S\boxdot\mathsf{J} \arrow[r, tail] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
(S\otimes S')\otimes S'' \arrow[r, two heads, "q_{S,S'}\otimes S''"] \arrow[d, "\alpha_{S,S',S''}"'] & (S\boxdot S')\otimes S'' \arrow[r, two heads, "q_{S\boxdot S',S''}"] & (S\boxdot S')\boxdot S'' \arrow[dddd, tail] \arrow[ddddll, dashed, "a_{S,S',S''}"'] \\
S\otimes(S'\otimes S'') \arrow[d, "S\otimes q_{S',S''}"'] & & \\
S\otimes(S'\boxdot S'') \arrow[d, two heads, "q_{S,S'\boxdot S''}"'] & & \\
S\boxdot(S'\boxdot S'') \arrow[rr, tail] & & T
\end{tikzcd}
$$

If λ is a natural isomorphism, then so is ℓ. This does not apply to ρ and α unless $\mathscr{E}$ is also closed under $S\otimes(-)$ for all $S \rightarrowtail T$.

Theorem 3.9. *Let T be a monoid in a skew monoidal category $\mathbb{C} = (\mathscr{C},\mathsf{I},\otimes)$, and let $(\mathscr{E},\mathscr{M})$ be a factorization system on $\mathscr{C}$. If $\mathscr{E}$ is closed under $(-)\otimes S$ for every $\mathscr{M}$-subobject $S \rightarrowtail T$, then $\mathscr{M}/\mathsf{T} = (\mathscr{M}/T,\mathsf{J},\boxdot)$ is a skew monoidal category. If $\mathbb{C}$ is left-normal, then so is $\mathscr{M}/\mathsf{T}$. If $\mathscr{E}$ is also closed under $S\otimes(-)$ for every $S \rightarrowtail T$ and $\mathbb{C}$ is monoidal, then $\mathscr{M}/\mathsf{T}$ is monoidal.*

Proof. See Appendix A. □

Remark 3.10. Closure of $\mathscr{E}$ under $T\otimes(-)$ does not in general imply closure of $\mathscr{E}$ under $S\otimes(-)$ for $S \rightarrowtail T$. Consider the factorization system $(\mathscr{E},\mathscr{M}) = (\text{componentwise surjective}, \text{componentwise full})$ on $[\mathbf{Poset},\mathbf{Poset}]$, with composition as $\otimes$. For an endofunctor $F : \mathbf{Poset} \to \mathbf{Poset}$, closure of $\mathscr{E}$ under $F\otimes(-)$ amounts to closure of **Epi** under F. The class **Epi** is closed under $V \Rightarrow (-)$ exactly when V is discrete. Hence, while this property holds for $\{0,1\} \Rightarrow (-)$, it does not hold for $\{0 \leq 1\} \Rightarrow (-) \rightarrowtail \{0,1\} \Rightarrow (-)$.

There are cases in which $\mathscr{E}$ is closed under $S\otimes(-)$ for all $S \rightarrowtail T$ even if $\mathscr{E}$ is not closed under $F\otimes(-)$ for general F: for example every $\mathscr{M}$-subobject of the endofunctor $M\times(-)$ on **Poset** has the form $S\times(-)$ for some $S \rightarrowtail M$, and functors of the form $S\times(-)$ send surjections to surjections.

Our task is now to show that the skew monoidal category $\mathscr{M}/\mathsf{T} = (\mathscr{M}/T,\mathsf{J},\boxdot)$ forms the canonical grading $(\mathscr{M}/\mathsf{T},\mathsf{T}_{\mathscr{M}},\mathrm{snd})$ of the monoid T when $\mathscr{E}$ is closed under $S\otimes(-)$ for each $S \rightarrowtail T$. The lax monoidal functor $\mathsf{T}_{\mathscr{M}} : \mathscr{M}/\mathsf{T} \to \mathbb{C}$ is given on objects by

$$T_{\mathscr{M}}(S, s : S \rightarrowtail T) = S$$

and has as unit and multiplication the $\mathscr{E}$-morphisms from the construction of J and $\boxdot$:

$$\mathsf{I} \xrightarrow{q} T_{\mathscr{M}}\mathsf{J} \qquad T_{\mathscr{M}}(S,s)\otimes T_{\mathscr{M}}(S',s') \xrightarrow{q_{S,S'}} T_{\mathscr{M}}((S,s)\boxdot(S',s'))$$

That this is lax monoidal is immediate from the definition of the structural morphisms of $\mathscr{M}/\mathsf{T}$. Finally, the monoidal natural transformation $\mathrm{snd}\mathscr{M}\mathsf{T} \Rightarrow \mathsf{T}$ is given by $\mathrm{snd}_{(S,s)} = s$. Monoidality of snd is immediate from the definitions of J and $\boxdot$. Hence $(\mathscr{M}/\mathsf{T},\mathsf{T}_{\mathscr{M}},\mathrm{snd})$ is a grading of T. Canonicity is the following theorem.

Theorem 3.11. *Let* T *be a monoid in a monoidal category* $(\mathscr{C}, \mathsf{I}, \otimes)$*, and let* $(\mathscr{E}, \mathscr{M})$ *be a factorization system on* $\mathscr{C}$ *such that* $\mathscr{E}$ *is closed under* $(-) \otimes S$ *for each* $\mathscr{M}$*-subobject* S *of* T*. The grading* $(\mathscr{M}/\mathsf{T}, \mathsf{T}_{\mathscr{M}}, \mathrm{snd})$ *is canonical in the sense that it is the pseudoterminal object of* $\mathbf{Grade}_{\mathscr{M}}\mathsf{T}$*. Explicitly, for every* $\mathscr{M}$*-grading* $(\mathscr{G}, \mathsf{G}, g)$ *of the monoid* T*, there is, up to isomorphism, a unique morphism* $(\mathsf{F}, f) : (\mathscr{G}, \mathsf{G}, g) \to (\mathscr{M}/\mathsf{T}, \mathsf{T}_{\mathscr{M}}, \mathrm{snd})$*, of* $\mathscr{M}$*-gradings of* T*.*

Proof. We have done most of the proof already as Theorem 2.3; we fill in the remaining parts. Recall from there that we define

$$Fd = (Gd, g_d) \quad Fh = Gh \qquad f_d = \mathrm{id}_{Gd}$$

We make F into a lax monoidal functor by using the unique diagonals of the following squares as the unit and multiplication (the squares commute because G is lax monoidal).

$$\begin{tikzcd}
\mathsf{I} \arrow[r, two heads, "q"] \arrow[d, "\eta"'] & \mathsf{J} \arrow[d, tail] \arrow[dl, dashed, "\eta"'] \\
GI \arrow[r, tail, "g_I"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
Gd \otimes Gd' \arrow[r, two heads, "q_{Gd,Gd'}"] \arrow[d, "\mu_{d,d'}"'] & Gd \boxdot Gd' \arrow[d, tail] \arrow[dl, dashed, "\mu_{d,d'}"'] \\
G(d \odot d') \arrow[r, tail, "g_{d \odot d'}"'] & T
\end{tikzcd}$$

This definition immediately implies that f is monoidal, and hence that (F, f) is a morphism of $\mathscr{M}$-gradings of T. Finally, given (F', f'), we show that $\beta_d = f'_d$ defines a isomorphism $\beta : (\mathsf{F}', f') \cong (\mathsf{F}, f)$. For this it remains to show that β is monoidal (it follows automatically that the inverse β^{-1} is monoidal). For compatibility with the multiplications, this amounts to showing that the square on the left below commutes. For this it is enough to show both paths in that square provide the unique diagonal of the square on the right, and this follows from the fact that f' is monoidal.

$$\begin{tikzcd}
F'd \boxdot F'd' \arrow[r, "f'_d \boxdot f'_{d'}"] \arrow[d, "\mu_{d,d'}"'] & Gd \boxdot Gd' \arrow[d, "\mu_{d,d'}"] \\
F'(d \odot d') \arrow[r, "f'_{d \odot d'}"'] & G(d \odot d')
\end{tikzcd}
\qquad
\begin{tikzcd}
F'd \otimes F'd' \arrow[r, two heads, "q_{F'd,F'd'}"] \arrow[d, "f'_d \otimes f'_{d'}"'] & F'd \boxdot F'd' \arrow[dd, tail] \arrow[ddl, dashed] \\
Gd \otimes Gd' \arrow[d, "\mu_{d,d'}"'] & \\
G(d \odot d') \arrow[r, tail, "g_{d \boxdot d'}"'] & T
\end{tikzcd}$$

Compatibility with the unit is similar. $\square$

Example 3.12. Let $(M, \varepsilon, \cdot)$ be a monoid in the cartesian monoidal category **Set**, and let T be the corresponding writer monad, which has endofunctor $TX = M \times X$, unit $\eta_X x = (\varepsilon, x)$, and multiplication $\mu_X(z, (z', x)) = (z \cdot z', x)$. Then T is a monoid in the monoidal category of endofunctors on **Set**, with functor composition. Consider the factorization system $(\mathscr{E}, \mathscr{M})$ on $[\mathbf{Set}, \mathbf{Set}]$ in which $\mathscr{E}$ (respectively $\mathscr{M}$) is componentwise surjective (resp. injective) natural transformations. The class $\mathscr{E}$ is closed under functor composition on both sides, so $\mathscr{M}/\mathsf{T}$ is monoidal and provides the canonical grading of T. We show in Example 2.6 that $\mathscr{M}$-subobjects of T are equivalently subsets $\Sigma \subseteq M$. Under this equivalence, the monoidal structure on $\mathscr{M}/\mathsf{T}$ is given by $\mathsf{J} = \{\varepsilon\}$ and $\Sigma \boxdot \Sigma' = \{z \cdot z' \mid z \in \Sigma, z' \in \Sigma'\}$. The graded monad $\mathsf{T}_{\mathscr{M}}$ is given by

$$T_{\mathscr{M}}\Sigma = \Sigma \times X \qquad \eta_X x = (\varepsilon, x) \qquad \mu_{\Sigma,\Sigma',X}(z, (z', x)) = (z \cdot z', x)$$

Example 3.13. We show in Example 2.8 that, when $\mathscr{M}$ is componentwise injective natural transformations and $T = V \Rightarrow V \times (-)$, the objects of $\mathscr{M}/T$ are equivalently subsets $\Sigma \subseteq (V \Rightarrow V) \times \mathrm{Equiv}_V$

satisfying a closure condition. When T is the state monad, these form a monoidal category $\mathscr{M}/\mathsf{T}$, and the graded monad $\mathsf{T}_{\mathscr{M}}$ has underlying functor

$$T_{\mathscr{M}}\Sigma X = \{f : V \to V \times X \mid \exists (p,R) \in \Sigma.\, p = \pi_1 \circ f \wedge (\pi_2 \circ f) \text{ respects } R\}$$

Example 3.5 provides another grading of T, in which the grades are subsets of $\{\mathsf{get},\mathsf{put}\}$. By Theorem 3.11, we obtain a morphism (F, f) of gradings. Under the characterization of grades as subsets Σ, the underlying functor F sends $e \subseteq \{\mathsf{get},\mathsf{put}\}$ to $Fe \subseteq (V \Rightarrow V) \times \mathrm{Equiv}_V$ as follows:

$$F\emptyset = \{(\mathrm{id}_V, V \times V)\} \qquad F\{\mathsf{get},\mathsf{put}\} = (V \Rightarrow V) \times \mathrm{Equiv}_V$$
$$F\{\mathsf{get}\} = \{(\mathrm{id}_V, R) \mid R \in \mathrm{Equiv}_V\} \qquad F\{\mathsf{put}\} = \{(p, V \times V) \mid p \text{ is a constant function or } \mathrm{id}_V\}$$

The underlying functor F sends a subset $e \subseteq \{\mathsf{get},\mathsf{put}\}$ to $Ge \rightarrowtail T$, as defined in Example 3.5.

4 Canonical grading by sets of shapes

When assigning grades to computations $t \in TX$, where T is a monad on **Set**, we are often interested only in the *shape* of the computation, which is $T!t \in T1$. In these cases, each grade e denotes a subset of the set $T1$ of shapes, and t has grade e when $T!t$ is in this set. More generally, if T is a monad on a category $\mathscr{D}$ with a terminal object 1, then the object of shapes is $T1$. Given a class $\mathscr{M}$ of morphisms of $\mathscr{D}$, we can consider grading by $\mathscr{M}$-subobjects of $T1$. We show in this section that these grades can be considered canonical, using a suitable class $\mathscr{M}'$ of morphisms of $[\mathscr{D},\mathscr{D}]$.

Definition 4.1. Let $\mathscr{A}$ be a category. If $\mathscr{A}$ has pullbacks, then a functor $F : \mathscr{A} \to \mathscr{D}$ is *cartesian* if it preserves pullbacks. A natural transformation $f : F \Rightarrow G : \mathscr{A} \to \mathscr{D}$ is *cartesian* if all of its naturality squares are pullbacks.

Lemma 4.2. *Let $(\mathscr{E},\mathscr{M})$ be a factorization system on a category $\mathscr{D}$ with pullbacks, and let $\mathscr{A}$ be a category with a terminal object. Then a factorization system $(\mathscr{E}',\mathscr{M}')$ on $[\mathscr{A},\mathscr{D}]$ is obtained by taking*

$$\mathscr{E}' = \text{natural transformations } e \text{ such that } e_1 \in \mathscr{E}$$
$$\mathscr{M}' = \text{cartesian natural transformations } m \text{ such that } m_1 \in \mathscr{M}$$

For every functor $G : \mathscr{A} \to \mathscr{D}$, there is an equivalence of categories $\mathscr{M}'/G \simeq \mathscr{M}/G1$.

Before giving the proof, we note that Kelly [6] considers $(\mathscr{E}',\mathscr{M}')$ in the case $(\mathscr{E},\mathscr{M}) = (\mathbf{Iso},\mathbf{Mor})$.

Proof. That $\mathscr{E}'$ and $\mathscr{M}'$ are closed under isomorphisms and composition is straightforward. To factorize a natural transformation $\tau : F \Rightarrow G$, we first factorize τ_1 using $(\mathscr{E},\mathscr{M})$, as on the bottom of the following diagram.

$$\begin{tikzcd}
FX \arrow[r, dashed, "e_X"] \arrow[d, "F!"'] \arrow[rr, bend left, "\tau_X"] & SX \arrow[r, "m_X"] \arrow[d] & GX \arrow[d, "G!"] \\
F1 \arrow[r, two heads, "\underline{e}"] \arrow[rr, bend right, "\tau_1"'] & \underline{S} \arrow[r, tail, "\underline{m}"] & G1
\end{tikzcd}$$

We then factorize any component τ_X as on the top, by taking as (SX, m_X) the pullback of $(\underline{S}, \underline{m})$ along $G!$, and taking as e_X the unique map from FX to this pullback. The objects S form a functor using unique maps into pullbacks, and e and m are natural transformations. When $X = 1$ we have $e_X \in \mathscr{E}$ and $m_X \in \mathscr{M}$ because the vertical morphisms in the diagram are all isomorphisms. Hence we have the required factorization of τ into $e \in \mathscr{E}'$ and $m \in \mathscr{M}'$. For orthogonality, unique diagonal fill-ins are constructed as unique maps to pullbacks.

The required equivalence of categories exists because every $\mathscr{M}'$-subobject $m : S \rightarrowtail G$ is determined up to isomorphism by the component m_1. The latter is the corresponding object of $\mathscr{M}/G1$. $\square$

The above lemma provides a construction of a factorization system $(\mathscr{E}', \mathscr{M}')$ in particular on endofunctor categories $[\mathscr{D}, \mathscr{D}]$, and in this case the $\mathscr{M}'$-subobjects of T are $\mathscr{M}$-subobjects of $T1$. However, we often do not have that $\mathscr{E}'$ is closed under functor composition on either side, and often do not get even a left-skew monoidal structure on $\mathscr{M}'/T$, as the following example demonstrates.

Example 4.3. Consider the factorization system $(\mathbf{Epi}, \mathbf{Mono})$ on $\mathscr{A} = \mathscr{D} = \mathbf{Set}$, and let T be the state monad for a set of states V:

$$T = V \Rightarrow V \times (-) \qquad \eta_X\, x\, v = (v, x) \qquad \mu_X\, f\, v = g\, v' \;\text{ where } (v', g) = f\, v$$

Since $T1 \cong V \Rightarrow V$, we have $\mathscr{M}'/T \simeq \mathbf{Mono}/(V \Rightarrow V)$, so that the canonical grades are equivalently subsets of $\Sigma \subseteq V \Rightarrow V$, ordered by inclusion. The subset Σ corresponds to the $\mathscr{M}'$-subobject $\mathbf{S}[\Sigma] \rightarrowtail T$ given by $\mathbf{S}[\Sigma]X = \{f : V \to V \times X \mid \pi_1 \circ f \in \Sigma\}$. Given such a subset, define $\mathrm{Cl}(\Sigma) = \{f : V \to V \times X \mid \forall v. \exists g \in \Sigma. \pi_1(f\,v) = g\,v\}$. The canonical grades Σ form a right-skew monoidal category with unit $\mathsf{J} = \{\mathrm{id}_V\}$ and tensor $\Sigma \boxdot \Sigma' = \Sigma \,\hat{\circ}\, \mathrm{Cl}(\Sigma')$, where $\Sigma \,\hat{\circ}\, \Sigma' = \{(f' \circ f) \mid f \in \Sigma, f' \in \Sigma'\}$. This right-skew monoidal structure is not left-normal, since $\mathsf{J} \boxdot \Sigma' = \mathrm{Cl}(\Sigma')$ is not in general equal to Σ'.

The factorization system $(\mathscr{E}', \mathscr{M}')$ is quite different from the componentwise lifting of $(\mathscr{E}, \mathscr{M})$ to $[\mathscr{A}, \mathscr{D}]$. But the next propositions show that the difference disappears if one restricts $[\mathscr{A}, \mathscr{D}]$ to cartesian natural transformations (and optionally further also to cartesian functors).

Lemma 4.4. *If $\mathscr{A}$ has pullbacks, $m : S \Rightarrow G : \mathscr{A} \to \mathscr{D}$ is a cartesian natural transformation, and G is cartesian, then S is also cartesian.*

Proof. Every pullback square in $\mathscr{A}$, as on the left below, induces a cube in $\mathscr{D}$, as on the right below. Four of the faces of this cube are pullbacks because m is cartesian, and the face on the right is a pullback because G is cartesian. It follows that the left face is also a pullback.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "f'"] \\
Y \arrow[r, "y"'] & Y'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
SX \arrow[rr, "m_X"] \arrow[dr, "Sx"] \arrow[dd, "Sf"'] & & GX \arrow[dr, "Gx"] \arrow[dd, "Gf" near start] & \\
& SX' \arrow[rr, "m_{X'}" near end] \arrow[dd, "Sf'" near start] & & GX' \arrow[dd, "Gf'"] \\
SY \arrow[rr, "m_Y" near end] \arrow[dr, "Sy"'] & & GY \arrow[dr, "Gy"] & \\
& SY' \arrow[rr, "m_{Y'}"'] & & GY'
\end{tikzcd}$$

$\square$

Lemma 4.5. *If $\mathscr{A}$ has pullbacks, then any factorization system on $[\mathscr{A}, \mathscr{D}]_{\mathrm{cartnt}}$ (all functors, but only cartesian natural transformations) restricts to a factorization system on $[\mathscr{A}, \mathscr{D}]_{\mathrm{cart}}$ (cartesian functors and cartesian natural transformations).*

Proof. We only need that, if $\tau : F \Rightarrow G$ with all of F, G, τ cartesian, factorizes as (S, e, m), then S is a cartesian functor. This is the case by the previous lemma since G and m are cartesian. $\square$

Definition 4.6. If $\mathscr{D}$ has pullbacks, then a factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathscr{D}$ is *stable* when $\mathscr{E}$ is closed under pullbacks along arbitrary morphisms.

(The analogous property for $\mathscr{M}$ is true in every factorization system.)

Lemma 4.7. *Assume that $\mathscr{D}$ has pullbacks, and let $(\mathscr{E},\mathscr{M})$ be a stable factorization system on $\mathscr{D}$. The componentwise lifting of $(\mathscr{E},\mathscr{M})$ to $[\mathscr{A},\mathscr{D}]$ restricts to a factorization system on $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$.*

Proof. We need to check that, if a cartesian natural transformation $\tau : F \Rightarrow G$ factorizes as (S,e,m) using $(\mathscr{E},\mathscr{M})$ componentwise, then e and m are cartesian natural transformations. Given any $f : X \to Y$, we can consider the naturality square of τ for f, which is by assumption cartesian. It breaks into naturality squares of e and m for f. We can then pull back (SY, m_Y) along Gf and be certain that the resulting morphism is in $\mathscr{M}$. The unique morphism from FX to the pullback vertex is a pullback of (FY, e_Y) along Sf by the pullback lemma, and is therefore in $\mathscr{E}$ by stability. We have obtained an isomorphic factorization of τ_X, hence the naturality squares of e and m are both pullbacks.

$$\begin{tikzcd}
FX \arrow[r, two heads, "e_X"] \arrow[rr, bend left, "\tau_X"] \arrow[dd, "Ff"'] \arrow[dr, dashed, two heads] & SX \arrow[r, tail, "m_X"] \arrow[dd, bend right, "Sf"'] & GX \arrow[dd, "Gf"] \\
& \bullet \arrow[ur, tail] \arrow[d] & \\
FY \arrow[r, two heads, "e_Y"] \arrow[rr, bend right, "\tau_Y"'] & SY \arrow[r, tail, "m_Y"] & GY
\end{tikzcd}$$

We also need to check that the diagonal fill-ins of cartesian squares built using $(\mathscr{E},\mathscr{M})$ componentwise are cartesian. This holds because the pullback lemma provides a two-out-of-three property for cartesian natural transformations: if $m \circ d$ and m are cartesian, then d is also cartesian. □

Lemma 4.8. *Let $(\mathscr{E},\mathscr{M})$ be any factorization system on $[\mathscr{A},\mathscr{D}]$. If all natural transformations in $\mathscr{M}$ are cartesian, then $(\mathscr{E},\mathscr{M})$ restricts to a factorization system on $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$.*

Proof. If $\tau = m \circ e$ and τ and m are cartesian, then e is cartesian by the pullback lemma. □

Lemma 4.9. *Let $(\mathscr{E},\mathscr{M})$ be a stable factorization system on a category $\mathscr{D}$ with pullbacks, and let $\mathscr{A}$ be a category with a terminal object. The factorization system $(\mathscr{E}',\mathscr{M}')$ on $[\mathscr{A},\mathscr{D}]$ from Lemma 4.2 and the componentwise lifting of $(\mathscr{E},\mathscr{M})$ to $[\mathscr{A},\mathscr{D}]$ both restrict to the same factorization system on $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$.*

Proof. By stability of $(\mathscr{E},\mathscr{M})$, for each cartesian natural transformation e, having $e_1 \in \mathscr{E}$ is equivalent to having $e_X \in \mathscr{E}$ for all X. Hence when restricted to $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$, the $\mathscr{E}'$-morphisms are exactly the $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$-morphisms whose components are in $\mathscr{E}$, and similarly for $\mathscr{M}'$. □

Example 4.10. Let T be the list monad on **Set**, so TX is the set of lists over X, the unit is $\eta_X\, x = [x]$, and the multiplication is $\mu_X [xs_1, \ldots, xs_n] = xs_1 \mathbin{+\!\!+} \cdots \mathbin{+\!\!+} xs_n$, where $(\mathbin{+\!\!+})$ is concatenation of lists. There is an isomorphism $T1 \cong \mathbb{N}$, so shapes are equivalently natural numbers (corresponding to the length of the list). Let $\mathscr{M}$ be the class of cartesian, componentwise injective natural transformations in $[\mathbf{Set},\mathbf{Set}]$. Then $\mathscr{M}$-subobjects of T are equivalently subsets $\Sigma \subseteq \mathbb{N}$. By the above, these form the canonical $\mathscr{M}$-grading of T. The monoidal structure on these subsets is given by

$$\mathsf{J} = \{1\} \qquad \Sigma \boxdot \Sigma' = \{\textstyle\sum_{i=1}^{n} m_i \mid n \in \Sigma, m_1, \ldots, m_n \in \Sigma'\}$$

The graded monad $\mathsf{T}_{\mathscr{M}}$ is given on objects by $\mathsf{T}_{\mathscr{M}} \Sigma X = \{xs \mid |xs| \in \Sigma\}$, where $|xs|$ is the length of the list xs.

5 Algebraic operations

In models of computational effects, we usually do not just want a (strong) monad T; we also want to equip T with a collection of *algebraic operations* in the sense of Plotkin and Power [12]. The latter provide interpretations of the constructs that cause the effects. When modelling computations using a graded monad, we similarly want algebraic operations for the graded monad; such a notion of algebraic operation was introduced in [5]. In this section, we therefore investigate the problem of constructing algebraic operations for the graded monad $\mathsf{T}_{\mathscr{M}}$, given algebraic operations for the monad T.

Throughout this section, we assume a monoidal category $\mathbb{C} = (\mathscr{C}, \otimes, \mathsf{I})$ that has finite products (for example, endofunctors on a category with finite products). When we write T^n below, we mean the product of n-many copies of T. We work only with normal (i.e., non-skew) monoidal categories in this section. The notion of algebraic operation for a graded monoid (e.g. a graded monad) that we use below works for monoidal categories, but the appropriate notion for skew monoidal categories would be more complicated. (It would use a list of grades e_i instead of a single grade e in the definition below.) Hence when we consider the canonical gradings below, we work under the assumption that they form a monoidal category (for example, when $\mathscr{E}$ is closed under $\otimes$ in both arguments).

The following definition generalizes the notion of algebraic operation for a monad to monoids.

Definition 5.1. Let $\mathsf{T} = (T, \eta, \mu)$ be a monoid in $\mathbb{C}$. An n-ary *algebraic operation* for T, where n is a natural number, is a morphism $\phi : T^n \to T$ such that

$$\begin{tikzcd}
T^n \otimes T \arrow[r, "\langle \pi_i \otimes T \rangle_i"] \arrow[d, "\phi \otimes T"'] & (T \otimes T)^n \arrow[r, "\mu^n"] & T^n \arrow[d, "\phi"] \\
T \otimes T \arrow[rr, "\mu"] & & T
\end{tikzcd}$$

Definition 5.2. Let $\mathsf{G} = (G, \eta, \mu) : \mathbb{G} \to \mathbb{C}$ be a $\mathbb{G}$-graded monoid in $\mathbb{C}$. A $(d_1, \ldots, d_n; d')$-ary algebraic operation for G, where $d_1, \ldots, d_n, d' \in \mathbb{G}$, is a natural transformation $\psi_e : \prod_i G(d_i \odot e) \Rightarrow G(d' \odot e)$ such that, for all $e, e' \in \mathscr{G}$,

$$\begin{tikzcd}
(\prod_i G(d_i \odot e)) \otimes Ge' \arrow[r, "\langle \pi_i \otimes Ge' \rangle_i"] \arrow[d, "\psi_e \otimes Ge'"'] & \prod_i (G(d_i \odot e) \otimes Ge') \arrow[r, "\prod_i \mu_{d_i \odot e, e'}"] & \prod_i G((d_i \odot e) \odot e') \arrow[r, "G\alpha"] & \prod_i G(d_i \odot (e \odot e')) \arrow[d, "\psi_{e \odot e'}"] \\
G(d' \odot e) \otimes Ge' \arrow[rr, "\mu_{d' \odot e, e'}"] & & G((d' \odot e) \odot e') \arrow[r, "G\alpha"] & G(d' \odot (e \odot e'))
\end{tikzcd}$$

Example 5.3. Let $\mathbb{C}$ be the cartesian monoidal category **Set**. Then an n-ary algebraic operation for a monoid T is a function $\phi : T^n \to T$ such that the multiplication of the monoid distributes over ϕ from the right. For example, if T is natural numbers with ordinary multiplication, then $\phi(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ is an n-ary algebraic operation.

Definition 5.4. Let $(\mathbb{G}, \mathsf{G}, g)$ be an $\mathscr{M}$-grading of a monoid T, where $\mathscr{M}$ is a class of morphisms in $\mathscr{C}$. We say that a $(d_1, \ldots, d_n; d')$-ary algebraic operation ψ for G is a *grading* of an n-ary algebraic operation ϕ for T when the following diagram commutes for all $e \in \mathbb{G}$.

$$\begin{tikzcd}
\prod_i G(d_i \odot e) \arrow[r, "\psi_e"] \arrow[d, "\prod_i g_{d_i \odot e}"'] & G(d' \odot e) \arrow[d, tail, "g_{d' \odot e}"] \\
T^n \arrow[r, "\phi"] & T
\end{tikzcd}$$

Suppose that T is a monoid in $\mathbb{C}$, and that $(\mathscr{E},\mathscr{M})$ is a factorization system on $\mathscr{C}$ such that $\mathscr{E}$ is closed under $(-)\otimes S$ for all $S \rightarrowtail T$. Then T has a canonical grading $\mathsf{T}_{\mathscr{M}} : \mathscr{M}/\mathsf{T} \to \mathbb{C}$ by Theorem 3.11. Suppose in addition that the skew-monoidal category $\mathscr{M}/\mathsf{T}$ is actually monoidal (which is the case when $\mathscr{E}$ is closed under $S \otimes (-)$ for all $S \rightarrowtail T$). We keep these assumptions without repeating them for the rest of this section.

Our goal in the rest of this section is to show that we can assign canonical grades to algebraic operations for T. To be more precise, let $\phi : T^n \to T$ is an n-ary algebraic operation for T, and let $R_1,\ldots,R_n$ be a list of grades ($\mathscr{M}$-subobjects of T). We show how to construct a grade R' and an algebraic operation

$$\psi : \textstyle\prod_i \mathsf{T}_{\mathscr{M}}(R_i \boxdot -) \to \mathsf{T}_{\mathscr{M}}(R' \boxdot -)$$

of arity $(R_1,\ldots,R_n;R')$ for $\mathsf{T}_{\mathscr{M}}$, that grades ϕ. The grade R' is in a sense canonical (see Theorem 5.6 below), and in fact every component of ψ is in $\mathscr{E}$.

To do this, we make the following two further assumptions about $\mathscr{E}$ for the rest of the section. Firstly that $\mathscr{E}$ contains the canonical morphisms $\langle \pi_i \otimes Y\rangle_i : (\prod_i X_i)\otimes Y \to \prod_i (X_i \otimes Y)$. This is the case in particular when $\otimes$ preserves finite products on the left (because $\mathscr{E}$ contains all isomorphisms); when $\otimes$ is composition of endofunctors this is automatically true. Secondly, we assume that $\mathscr{E}$ is closed under finite products, i.e. that $\prod_i e_i : \prod_i X_i \to \prod_i Y_i$ is in $\mathscr{E}$ whenever all of the morphisms $e_i : X_i \twoheadrightarrow Y_i$ are in $\mathscr{E}$. This is the case for all of the factorization systems we consider above.

The key lemma that enables us to construct ψ is the following, which characterizes algebraic operations for the canonical grading $\mathsf{T}_{\mathscr{M}}$ of T.

Lemma 5.5. *Let $\phi : T^n \to T$ be an n-ary algebraic operation for T, and let $R_1,\ldots,R_n,R'$ be $\mathscr{M}$-subobjects of T. There is a bijection between (1) morphisms $p : \prod_i R_i \to R'$ such that*

$$\begin{tikzcd}
\prod_i R_i \arrow[r, "p"] \arrow[d, tail] & R' \arrow[d, tail] \\
T^n \arrow[r, "\phi"] & T
\end{tikzcd}$$

and (2) $(R_1,\ldots,R_n;R')$-ary algebraic operations ψ for $\mathsf{T}_{\mathscr{M}}$ that grade ϕ.

Proof. Given a morphism p as in (1), the following square commutes because ϕ is algebraic, and the square hence has a unique diagonal ψ_S. Further applications of orthogonality show that ψ is an algebraic operation. It is a grading of ϕ by definition.

$$\begin{tikzcd}
(\prod_i R_i)\otimes S \arrow[r, two heads, "\langle \pi_i \otimes S\rangle_i"] \arrow[d, "p\otimes R'"'] & \prod_i (R_i \otimes S) \arrow[r, two heads, "\prod_i q_{R_i,S}"] & \prod_i (R_i \boxdot S) \arrow[d, tail] \arrow[ddll, dashed, "\psi_S"'] \\
R' \otimes S \arrow[d, two heads, "q_{R',S}"'] & & T^n \arrow[d, "\phi"] \\
R' \boxdot S \arrow[rr, tail] & & T
\end{tikzcd}$$

In the other direction, given ψ we have a morphism p as follows; this p makes the diagram required for (1) commute because ψ is a grading of ϕ.

$$p : \textstyle\prod_i R_i \xrightarrow{\prod_i \rho_{R_i}} \prod_i (R_i \boxdot \mathsf{J}) \xrightarrow{\psi_{\mathsf{J}}} R' \boxdot \mathsf{J} \xrightarrow{\rho_{R'}^{-1}} R'$$

From algebraicity of ψ it follows that this p makes the upper triangle of the above square commute and hence, by uniqueness of the diagonal, that ψ is the only grading of ϕ that induces this p. The construction of p from ψ is therefore injective. The construction of ψ from p is injective because, if the upper triangle of the above diagram commutes then so does the following, hence p is determined by ψ as above.

$$\begin{tikzcd}
\prod_i R_i \arrow[dr, "(\prod_i \otimes q)\circ\rho"] \arrow[rrrd, bend left, "\prod_i \rho_{R_i}"] \arrow[ddd, "p"'] & & & \\
& (\prod_i R_i)\otimes \mathsf{J} \arrow[r, "\langle \pi_i \otimes \mathsf{J}\rangle_i"] \arrow[d, "p\otimes \mathsf{J}"'] & \prod_i (R_i \otimes \mathsf{J}) \arrow[r, "\prod_i q_{R_i,\mathsf{J}}"] & \prod_i (R_i \boxdot \mathsf{J}) \arrow[d, "\phi_{\mathsf{J}}"] \\
& R'\otimes \mathsf{J} \arrow[rr, "q_{R',\mathsf{J}}"] & & R' \boxdot \mathsf{J} \\
R' \arrow[ur, "(R'\otimes q)\circ\rho"] \arrow[rrru, bend right, "\rho_{R'}"'] & & &
\end{tikzcd}$$

□

Now given an n-ary algebraic operation ϕ for T and a fixed tuple $R_1,\ldots,R_n$ of $\mathscr{M}$-subobjects of T, we construct the canonical R' by factorizing $\prod_i R_i \rightarrowtail T^n \xrightarrow{\phi} T$ as $\prod_i R_i \xrightarrow{p} R' \rightarrowtail T$. The preceding lemma then provides us with an $(R_1,\ldots,R_n;R')$-algebraic operation ψ for $\mathsf{T}_{\mathscr{M}}$.

Theorem 5.6. *Let $\phi : T^n \to T$ be an n-ary algebraic operation for T.*

1. *The construction defines an $(R_1,\ldots,R_n;R')$-ary algebraic operation ψ for $\mathsf{T}_{\mathscr{M}}$, and ψ grades ϕ. Every component ψ_S is in $\mathscr{E}$.*
2. *For any $\mathscr{M}$-subobject $R'' \rightarrowtail T$ and $(R_1,\ldots,R_n;R'')$-ary algebraic operation ψ' for $\mathsf{T}_{\mathscr{M}}$, such that ψ' grades ϕ, there is a unique $f : R' \to R''$ in $\mathscr{M}/T$ such that $(f \boxdot S)\circ\psi_S = \psi'_S$ for all S.*

Proof. The first sentence of (1) is immediate from Lemma 5.5. Each ψ_S is in $\mathscr{E}$ because we have $\psi_S \circ e = e'$ for some $e, e' \in \mathscr{E}$ (this is the upper triangle in the definition of ψ_S, using the fact that p is in $\mathscr{E}$). This implies $\psi_S \in \mathscr{E}$ because $\mathscr{E}$-morphisms satisfy a two-out-of-three property. For (2), given ψ', we obtain from Lemma 5.5 a morphism $p' : \prod_i R_i \to R''$ making the diagram on the left below commute.

$$\begin{tikzcd}
\prod_i R_i \arrow[r, "p'"] \arrow[d, tail] & R'' \arrow[d, tail] \\
T^n \arrow[r, "\phi"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod_i R_i \arrow[r, two heads, "p"] \arrow[d, "p'"'] & R' \arrow[d, tail] \arrow[dl, dashed, "f"] \\
R'' \arrow[r, tail] & T
\end{tikzcd}$$

For a morphism $f : R' \to R''$ in $\mathscr{M}/T$, the condition that $(f \boxdot S)\circ\psi_S = \psi'_S$ for all S implies (using $S = \mathsf{J}$) that $f \circ p = p'$. The converse also holds, using orthogonality. Hence the conditions on the morphism f are equivalent to commutativity of the square on the right above. The outside of the square commutes and p is in $\mathscr{E}$, so there exists a unique f. □

Example 5.7. Consider the writer monad given by $T = M \times (-)$ from Example 3.12. Every $z \in M$ induces a unary algebraic operation $\phi_z : T \to T$, defined by $\phi_{z,X}(z',x) = (z\cdot z', x)$. When $\mathscr{M}$ is the class of componentwise injective natural transformations, the canonical $\mathscr{M}$-grading of T has subsets $\Sigma \subseteq M$ as grades, and $\mathsf{T}_{\mathscr{M}}\Sigma = \Sigma \times (-)$. Every input grade $P \subseteq M$ induces a canonical output grade $P'_z \subseteq M$ and algebraic operation $\psi_{z,\Sigma} : \mathsf{T}_{\mathscr{M}}(P \boxdot \Sigma) \Rightarrow \mathsf{T}_{\mathscr{M}}(P'_z \boxdot \Sigma)$, and these turn out to be:

$$P'_z = \{z \cdot z' \mid z' \in P\} \qquad \psi_{z,\Sigma,X}(z',x) = (z'\cdot z, x)$$

Example 5.8. Let T be the list monad on **Set**. This has a binary algebraic operation $(+\!\!+) : T \times T \Rightarrow T$ that concatenates a pair of lists. As we explain in Example 4.10, subsets $\Sigma \subseteq \mathbb{N}$ provide a canonical grading of T. If P_1, P_2 are subsets of $\mathbb{N}$, then the grade we construct for the algebraic operation $(+\!\!+)$ as above is $P' = \{n_1 + n_2 \mid n_1 \in P_1, n_2 \in P_2\}$, and the algebraic operation for $\mathsf{T}_{\mathscr{M}}$ is the natural transformation $\mathsf{T}_{\mathscr{M}}(P_1 \boxdot -) \times \mathsf{T}_{\mathscr{M}}(P_2 \boxdot -) \Rightarrow \mathsf{T}_{\mathscr{M}}(P' \boxdot -)$ that maps (xs_1, xs_2) to $xs_1 +\!\!+ xs_2$.

6 Conclusion and future work

We have demonstrated that factorization systems provide a unifying framework for grading of monads with subfunctors, in fact, monoids with subobjects. Skew monoidal categories turn out to be a more robust setting for this than monoidal categories, which means, among other things, that this framework will be directly applicable also to relative monads.

The abstract framework is pleasingly elegant, but for applications we would like obtain a stronger intuition for its reach. We intend to explore this first by working out the canonical gradings with (strong) subfunctors of further standard example (strong) monads from programming semantics, for the factorization systems considered in this paper and possibly others. Indeed, the examples may point to further factorization systems of interest. The outcomes of this exploration will hopefully lead to some new heuristics for the construction of graded monads for applications such as type-and-effect systems.

Programming semantics applications also suggest trying grading with subfunctors on (strong) lax monoidal functors ("applicative functors") and (strong) monads in **Prof** ("arrows"). Comonads can be graded with quotient functors.

Acknowledgements D.M. and T.U. were supported by the Icelandic Research Fund grants 196323-053 and 228684-051. This work started during T.U.'s visit to F.B. at LIPN as an invited professor in Sept. 2019, but was mostly stalled during the Covid pandemic.

References

[1] Tobias Fritz & Paolo Perrone (2018): *A Criterion for Kan Extensions of Lax Monoidal Functors*. arXiv eprint 1809.10481 [math.CT], doi:10.48550/arXiv.1809.10481.

[2] Soichiro Fujii, Shin-ya Katsumata & Paul-André Melliès (2016): *Towards a Formal Theory of Graded Monads*. In Bart Jacobs & Christof Löding, editors: *Proc. of 19th Int. Conf. on Foundations of Software Science and Computation Structures, FoSSaCS 2016*, *Lect. Notes in Comput. Sci.* 9634, Springer, Cham, pp. 513–530, doi:10.1007/978-3-662-49630-5_30.

[3] Ohad Kammar & Dylan McDermott (2018): *Factorisation Systems for Logical Relations and Monadic Lifting in Type-and-Effect System Semantics*. *Electron. Notes Theor. Comput. Sci.* 341, pp. 239–260, doi:10.1016/j.entcs.2018.11.012.

[4] Shin-ya Katsumata (2014): *Parametric Effect Monads and Semantics of Effect Systems*. In: *Proc. of 41st Ann. ACM SIGPLAN-SIGACT Symp. on Principles of Programming Languages, POPL '14*, ACM, New York, pp. 633–646, doi:10.1145/2535838.2535846.

[5] Shin-ya Katsumata, Dylan McDermott, Tarmo Uustalu & Nicolas Wu (2022): *Flexible Presentations of Graded Monads*. *Proc. ACM Program. Lang.* 6(ICFP). To appear.

[6] G. Max Kelly (1992): *On Clubs and Data-Type Constructors*. In Michael P. Fourman, Peter T. Johnstone & Andrew M. Pitts, editors: *Applications of Categories in Computer Science*, *London. Math. Soc. Lect. Note Series* 177, Cambridge Univ. Press, pp. 163–190, doi:10.1017/cbo9780511525902.010.

[7] Stephen Lack & Ross Street (2014): *Triangulations, Orientals, and Skew Monoidal Categories*. *Adv. Math.* 258, pp. 351–396, doi:10.1016/j.aim.2014.03.003.

[8] John M. Lucassen & David K. Gifford (1988): *Polymorphic Effect Systems*. In: *Proc. of 15th ACM SIGPLAN-SIGACT Symp. on Principles of Programming Languages, POPL '88*, ACM, New York, pp. 47–57, doi:10.1145/73560.73564.

[9] Dylan McDermott, Maciej Piróg & Tarmo Uustalu (2020): *Degrading Lists*. In: *Proc. of 22nd Int. Symp. on Principles and Practice of Declarative Programming, PPDP '20*, ACM, New York, pp. 6:1–6:14, doi:10.1145/3414080.3414084.

[10] Paul-André Melliès (2012): *Parametric Monads and Enriched Adjunctions*. Manuscript. Available at `https://www.irif.fr/~mellies/tensorial-logic/8-parametric-monads-and-enriched-adjunctions.pdf`.

[11] Eugenio Moggi (1989): *Computational Lambda-Calculus and Monads*. In: *Proc. of 4th Ann. Symp. on Logic in Computer Science, LICS '89*, IEEE Press, Los Alamitos, CA, pp. 14–23, doi:10.1109/lics.1989.39155.

[12] Gordon Plotkin & John Power (2003): *Algebraic Operations and Generic Effects*. *Appl. Categ. Struct.* 11, pp. 69–94, doi:10.1023/a:1023064908962.

[13] Alexander Smirnov (2008): *Graded Monads and Rings of Polynomials*. *J. Math. Sci.* 151, pp. 3032–3051, doi:10.1007/s10958-008-9013-7.

[14] Kornel Szlachányi (2012): *Skew-Monoidal Categories and Bialgebroids*. *Adv. Math.* 231(3–4), pp. 1694–1730, doi:10.1016/j.aim.2012.06.027.

A Proof of Theorem 3.9

Given a monoidal category $(\mathscr{C}, \mathsf{I}, \otimes, \lambda, \rho, \alpha)$ with a monoid object (T, η, μ) and an orthogonal factorization system $(\mathscr{E}, \mathscr{M})$. We assume that $\mathscr{E}$ is closed under $(-) \otimes X$ for all $(X, x) \in \mathscr{M}/T$.

Our aim is to show $\mathscr{M}/T$ carries a left-skew monoidal category structure $((\mathsf{J}, j), \boxdot, \ell, r, a)$.

The unit (J, j) and tensor $(X \boxdot Y, x \boxdot y)$ of two objects (X, x), (Y, y) are defined as the factorizations shown in the diagrams below.

$$\begin{tikzcd}
\mathsf{I} \arrow[rr, "\eta"] \arrow[dr, two heads, "q"'] & & T \\
& \mathsf{J} \arrow[ur, tail, "j"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r, "x \otimes y"] \arrow[dr, two heads, "q_{x,y}"'] & T \otimes T \arrow[r, "\mu"] & T \\
& X \boxdot Y \arrow[ur, tail, "x \boxdot y"'] &
\end{tikzcd}$$

The functorial action of $\boxdot$ on two morphisms $f : (X, x) \to (X', x')$ and $g : (Y, y) \to (Y', y')$ is a morphism $f \boxdot g : (X \boxdot Y, x \boxdot y) \to (X' \boxdot Y', x' \boxdot y')$ defined as the diagonal fill-in of the commuting square below.

$$\begin{tikzcd}
X \otimes Y \arrow[r, two heads, "q_{x,y}"] \arrow[dr, "x \otimes y"] \arrow[dd, "f \otimes g"'] & X \boxdot Y \arrow[drr, tail, "x \boxdot y"] \arrow[dd, dashed, tail, "f \boxdot g"] & & \\
& T \otimes T \arrow[rr, "\mu"] & & T \\
X' \otimes Y' \arrow[r, two heads, "q_{x',y'}"'] \arrow[ur, "x' \otimes y'"'] & X' \boxdot Y' \arrow[urr, tail, "x' \boxdot y'"'] & &
\end{tikzcd}$$

The left unitor ℓ and associator a are also defined as the diagonal fill-ins for suitable commuting squares. The right unitor is just a composition of morphisms.

Definition of ℓ:

$$\begin{tikzcd}
\mathsf{I} \otimes X \arrow[r, two heads, "q \otimes X"] \arrow[dr, "\mathsf{I} \otimes x"'] \arrow[dd, "\lambda_X"'] & \mathsf{J} \otimes X \arrow[r, two heads, "q_{j,x}"] \arrow[dr, "j \otimes x"] & \mathsf{J} \boxdot X \arrow[dr, tail, "j \boxdot x"] \arrow[ddll, dashed, tail, "\ell_x"] & \\
& \mathsf{I} \otimes T \arrow[r, "\eta \otimes T"] \arrow[d, "\lambda_T"] & T \otimes T \arrow[r, "\mu"] & T \\
X \arrow[r, tail, "x"'] & T \arrow[urr, equal] & &
\end{tikzcd}$$

Definition of r:

$$\begin{tikzcd}
X \arrow[r, tail, "x"] \arrow[dd, "\rho_X"'] \arrow[ddrr, dashed, tail, "r_x"] & T \arrow[d, "\rho_T"] \arrow[drr, equal] & & \\
& T \otimes \mathsf{I} \arrow[r, "T \otimes \eta"] & T \otimes T \arrow[r, "\mu"] & T \\
X \otimes \mathsf{I} \arrow[r, two heads, "X \otimes q"'] \arrow[ur, "x \otimes \mathsf{I}"] & X \otimes \mathsf{J} \arrow[r, two heads, "q_{x,j}"'] \arrow[ur, "x \otimes j"] & X \boxdot \mathsf{J} \arrow[ur, tail, "x \boxdot j"'] &
\end{tikzcd}$$

Definition of a:

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, two heads, "q_{x,y}\otimes Z"] \arrow[dr, "(x\otimes y)\otimes z"'] \arrow[dddd, "\alpha_{X,Y,Z}"'] & (X\boxdot Y)\otimes Z \arrow[r, two heads, "q_{x\boxdot y,z}"] \arrow[dr, "(x\boxdot y)\otimes z"] & (X\boxdot Y)\boxdot Z) \arrow[dddd, dashed, "a_{x,y,z}"] \arrow[ddr, tail, "(x\boxdot y)\boxdot z"] & \\
& (T\otimes T)\otimes T \arrow[r, "\mu\otimes T"'] \arrow[dd, "\alpha_{T,T,T}"'] & T\otimes T \arrow[dr, "\mu"] & \\
& & & T \\
& T\otimes(T\otimes T) \arrow[r, "T\otimes\mu"] & T\otimes T \arrow[ur, "\mu"'] & \\
X\otimes(Y\otimes Z) \arrow[r, "X\otimes q_{y,z}"'] \arrow[ur, "x\otimes(y\otimes z)"] & X\otimes(Y\boxdot Z) \arrow[r, two heads, "q_{x,y\boxdot z}"'] \arrow[ur, "x\otimes(y\boxdot z)"'] & X\boxdot(Y\boxdot Z) \arrow[uur, tail, "x\boxdot(y\boxdot z)"'] &
\end{tikzcd}$$

The proofs of functoriality of $\boxdot$ and naturality of ℓ, r and a are easy and omitted.

The equations (m1)–(m5) for ℓ, r, a are each proved by the diagram chases below from the respective equations of λ, ρ, α using the properties of $\boxdot$, ℓ, r, a arising from their construction (the two triangles that the fill-in breaks the commuting square into commute). This is done by showing that the precompositions of the left-hand and right-hand sides with a suitable $\mathscr{E}$-morphism are equal.

Proof of (m1):

$$\begin{tikzcd}
\mathsf{I} \arrow[r, two heads, "q"] \arrow[dr, "\rho_{\mathsf{I}}"'] \arrow[dd, equal] & \mathsf{J} \arrow[rr, tail, "\rho_{\mathsf{J}}"'] \arrow[rrrr, tail, bend left, "r_j"] & & \mathsf{J}\otimes\mathsf{I} \arrow[dr, "\mathsf{J}\otimes q"] & & \\
& & \mathsf{I}\otimes\mathsf{I} \arrow[ur, two heads, "q\otimes\mathsf{I}"'] \arrow[dl, "\lambda_{\mathsf{I}}"'] \arrow[dr, "\mathsf{I}\otimes q"] & & \mathsf{J}\otimes\mathsf{J} \arrow[r, two heads, "q_{j,j}"'] & \mathsf{J}\boxdot\mathsf{J} \arrow[dllll, tail, bend left, "\ell_j"] \\
\mathsf{I} \arrow[r, two heads, "q"] & \mathsf{J} & & \mathsf{I}\otimes\mathsf{J} \arrow[ll, "\lambda_{\mathsf{J}}"'] \arrow[ur, two heads, "q\otimes\mathsf{J}"'] & &
\end{tikzcd}$$

Proof of (m2):

$$\begin{tikzcd}
& (\mathsf{J}\otimes Y)\otimes Z \arrow[r, two heads, "q_{j,y}\otimes Z"] \arrow[d, "\alpha_{\mathsf{J},Y,Z}"] & (\mathsf{J}\boxdot Y)\otimes Z \arrow[r, two heads, "q_{j\boxdot y,z}"] & (\mathsf{J}\boxdot Y)\boxdot Z \arrow[d, tail, "a_{j,y,z}"] \\
& \mathsf{J}\otimes(Y\otimes Z) \arrow[r, "\mathsf{J}\otimes q_{y,z}"] & \mathsf{J}\otimes(Y\boxdot Z) \arrow[r, two heads, "q_{j,y\boxdot z}"] & \mathsf{J}\boxdot(Y\boxdot Z) \arrow[dd, tail, "\ell_{y\boxdot z}"] \\
(\mathsf{I}\otimes Y)\otimes Z \arrow[uur, two heads, "(q\otimes Y)\otimes Z"] \arrow[r, "\alpha_{\mathsf{I},Y,Z}"] \arrow[dr, "\lambda_Y\otimes Z"] \arrow[ddr, two heads, "(q\otimes Y)\otimes Z"'] & \mathsf{I}\otimes(Y\otimes Z) \arrow[u, two heads, "q\otimes(Y\otimes Z)"'] \arrow[r, "\mathsf{I}\otimes q_{y,z}"] \arrow[d, "\lambda_{Y\otimes Z}"] & \mathsf{I}\otimes(Y\boxdot Z) \arrow[u, two heads, "q\otimes(Y\boxdot Z)"'] \arrow[dr, "\lambda_{Y\boxdot Z}"] & \\
& Y\otimes Z \arrow[rr, two heads, "q_{y,z}"] & & Y\boxdot Z \\
& (\mathsf{J}\otimes Y)\otimes Z \arrow[r, two heads, "q_{j,y}\otimes Z"'] & (\mathsf{J}\boxdot Y)\otimes Z \arrow[ul, "\ell_y\otimes Z"'] \arrow[r, two heads, bend left, "q_{y\circ\ell_y,z}"] \arrow[r, two heads, bend right, "q_{j\boxdot y,z}"'] & (\mathsf{J}\boxdot Y)\boxdot Z \arrow[u, tail, "\ell_y\boxdot Z"']
\end{tikzcd}$$

Proof of (m3):

$$
\begin{tikzcd}
 & X \boxdot Y \arrow[r, "\rho_{X \boxdot Y}"] \arrow[rrr, tail, bend left, "r_{x \boxdot y}"] & (X \boxdot Y) \otimes \mathsf{I} \arrow[r, "(X \boxdot Y) \otimes q"] & (X \boxdot Y) \otimes \mathsf{J} \arrow[r, two heads, "q_{x \boxdot y, j}"] & (X \boxdot Y) \boxdot \mathsf{J} \arrow[dddd, tail, "a_{x,y,j}"] \\
 & (X \otimes Y) \otimes \mathsf{I} \arrow[ur, "q_{x,y} \otimes \mathsf{I}"] \arrow[r, "(X \otimes Y) \otimes q"] \arrow[dd, "\alpha_{X,Y,\mathsf{I}}"] & (X \otimes Y) \otimes \mathsf{J} \arrow[ur, "q_{x,y} \otimes \mathsf{J}"'] \arrow[dd, "\alpha_{X,Y,\mathsf{J}}"] & & \\
X \otimes Y \arrow[uur, two heads, "q_{x,y}"] \arrow[ur, "\rho_{X \otimes Y}"'] \arrow[dr, "X \otimes \rho_Y"] \arrow[ddr, two heads, bend right, "q_{x,y}"'] \arrow[ddr, two heads, bend left, "q_{x,y \boxdot j} \circ q_{y,j} \circ Y \otimes q \circ \rho_Y"] & & & & \\
 & X \otimes (Y \otimes \mathsf{I}) \arrow[r, "X \otimes (Y \otimes q)"] & X \otimes (Y \otimes \mathsf{J}) \arrow[r, "X \otimes q_{y,j}"] & X \otimes (Y \boxdot \mathsf{J}) \arrow[dr, two heads, "q_{x, y \boxdot j}"] & \\
 & X \boxdot Y \arrow[r, "X \boxdot \rho_Y"'] \arrow[rrr, bend right, "X \boxdot r_y"'] & X \boxdot (Y \otimes \mathsf{I}) \arrow[r, "X \boxdot (Y \otimes q)"'] & X \boxdot (Y \otimes \mathsf{J}) \arrow[r, "X \boxdot q_{y,j}"'] & X \boxdot (Y \boxdot \mathsf{J})
\end{tikzcd}
$$

Proof of (m4):

$$
\begin{tikzcd}
 & X \boxdot Z \arrow[r, "\rho_X \boxdot Z"] \arrow[rrr, bend left, "r_x \boxdot Z"] & (X \otimes \mathsf{I}) \boxdot Z \arrow[r, "(X \otimes q) \boxdot Z"] & (X \otimes \mathsf{J}) \boxdot Z \arrow[r, two heads, "q_{x,j} \boxdot Z"] & (X \boxdot \mathsf{J}) \boxdot Z \arrow[ddd, tail, "a_{x,j,z}"] \\
X \otimes Z \arrow[ur, two heads, bend left, "q_{x,z}"] \arrow[ur, two heads, bend right, "q_{x \boxdot j} \circ q_{x,j} \circ X \otimes q \circ \rho_{X},z"'] \arrow[r, "\rho_X \otimes Z"'] \arrow[d, equal] & (X \otimes \mathsf{I}) \otimes Z \arrow[r, "(X \otimes q) \otimes Z"] \arrow[d, "\alpha_{X,\mathsf{I},Z}"] & (X \otimes \mathsf{J}) \otimes Z \arrow[r, two heads, "q_{x,j} \otimes Z"] \arrow[d, "\alpha_{X,\mathsf{J},Z}"'] & (X \boxdot \mathsf{J}) \otimes Z \arrow[ur, "q_{x \boxdot j, z}"] & \\
X \otimes Z \arrow[dr, two heads, "q_{x,z}"'] & X \otimes (\mathsf{I} \otimes Z) \arrow[l, "X \otimes \lambda_Z"'] \arrow[r, "X \otimes (q \otimes Z)"] \arrow[dr, two heads, bend right, "q_{x,z} \circ \lambda_Z"'] \arrow[dr, two heads, bend left, "q_{x, j \boxdot z} \circ q_{j,z} \circ q \otimes Z"] & X \otimes (\mathsf{J} \otimes Z) \arrow[r, "X \otimes q_{j,z}"] & X \otimes (\mathsf{J} \boxdot Z) \arrow[dr, two heads, "q_{x, j \boxdot Z}"] & \\
 & X \boxdot Z & X \boxdot (\mathsf{I} \otimes Z) \arrow[l, "X \boxdot \lambda_Z"'] \arrow[r, "X \boxdot (q \otimes Z)"] & X \boxdot (\mathsf{J} \otimes Z) \arrow[r, "X \boxdot q_{j,z}"] & X \boxdot (\mathsf{J} \boxdot Z) \arrow[lll, bend left, "X \boxdot \ell_z"]
\end{tikzcd}
$$

Proof of (m5):

$$
\begin{tikzcd}
 & ((X\boxdot Y)\otimes Z)\otimes W \arrow[r, two heads, "q_{x\boxdot y,z}\otimes W"] \arrow[dr, "\alpha_{X\boxdot Y,Z,W}"] & ((X\boxdot Y)\boxdot Z)\otimes W \arrow[r, two heads, "q_{(x\boxdot y)\boxdot z,w}"] & ((X\boxdot Y)\boxdot Z)\boxdot W \arrow[dr, tail, "a_{x\boxdot y,z,w}"] & \\
 & (X\otimes Y)\otimes (Z\otimes W) \arrow[r, two heads, "q_{x,y}\otimes(Z\otimes W)"] \arrow[dr, "(X\otimes Y)\otimes q_{z,w}"'] \arrow[ddd, "\alpha_{X,Y,Z\otimes W}"] & (X\boxdot Y)\otimes (Z\otimes W) \arrow[r, two heads, "(X\boxdot Y)\otimes q_{z,w}"] & (X\boxdot Y)\otimes (Z\boxdot W) \arrow[r, two heads, "q_{x\boxdot y,z\boxdot w}"] & (X\boxdot Y)\boxdot (Z\boxdot W) \arrow[ddd, tail, "a_{x,y,z\boxdot w}"] \\
 & & (X\otimes Y)\otimes (Z\boxdot W) \arrow[ur, two heads, "q_{x,y}\otimes(Z\boxdot W)"'] \arrow[d, "\alpha_{X,Y,Z\boxdot W}"] & & \\
((X\otimes Y)\otimes Z)\otimes W \arrow[uuur, two heads, "(q_{x,y}\otimes Z)\otimes W"] \arrow[uur, "\alpha_{X\otimes Y,Z,W}"'] \arrow[ddr, "\alpha_{X,Y,Z}\otimes W"] \arrow[dddr, two heads, "(q_{x,y}\otimes Z)\otimes W"'] & X\otimes(Y\otimes(Z\otimes W)) \arrow[r, "X\otimes(Y\otimes q_{z,w})"] & X\otimes(Y\otimes(Z\boxdot W)) \arrow[r, "X\otimes q_{y,z\boxdot w}"] & X\otimes(Y\boxdot(Z\boxdot W)) \arrow[r, two heads, "q_{x,y\boxdot(z\boxdot w)}"] & X\boxdot(Y\boxdot(Z\boxdot W)) \\
 & X\otimes((Y\otimes Z)\otimes W) \arrow[u, "X\otimes\alpha_{Y,Z,W}"'] \arrow[r, "X\otimes(q_{y,z}\otimes W)"] & X\otimes((Y\boxdot Z)\otimes W) \arrow[r, "X\otimes q_{y\boxdot z,w}"] & X\otimes((Y\boxdot Z)\boxdot W) \arrow[u, "X\otimes a_{y,z,w}"] \arrow[r, two heads, "q_{x,(y\boxdot z)\boxdot w}"] & X\boxdot((Y\boxdot Z)\boxdot W) \arrow[u, tail, "X\boxdot a_{y,z,w}"'] \\
 & (X\otimes(Y\otimes Z))\otimes W \arrow[u, "\alpha_{X,Y\otimes Z,W}"'] \arrow[r, "(X\otimes q_{y,z})\otimes W"] & (X\otimes(Y\boxdot Z))\otimes W \arrow[u, "\alpha_{X,Y\boxdot Z,W}"'] \arrow[r, two heads, "q_{x,y\boxdot z}\otimes W"] & (X\boxdot(Y\boxdot Z))\otimes W \arrow[r, two heads, "q_{x\boxdot(y\boxdot z),w}"] & (X\boxdot(Y\boxdot Z))\boxdot W \arrow[u, tail, "a_{x,y\boxdot z,w}"'] \\
 & ((X\boxdot Y)\otimes Z)\otimes W \arrow[r, two heads, "q_{x\boxdot y,z}\otimes W"'] & ((X\boxdot Y)\boxdot Z)\otimes W \arrow[ur, tail, "a_{x,y,z}\otimes W"] \arrow[r, two heads, "q_y(x\boxdot y)\boxdot z,w"'] & ((X\boxdot Y)\boxdot Z)\boxdot W \arrow[ur, tail, "a_{x,y,z}\boxdot W"'] &
\end{tikzcd}
$$