Canonical Gradings of Monads

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We define a notion of grading of a monoid T in a monoidal category $\mathscr{C}$, relative to a class of morphisms $\mathscr{M}$ (which provide a notion of $\mathscr{M}$-subobject). We show that, under reasonable conditions (including that $\mathscr{M}$ forms a factorization system), there is a canonical grading of T. Our application is to graded monads and models of computational effects. We demonstrate our results by characterizing the canonical gradings of a number of monads, for which $\mathscr{C}$ is endofunctors with composition. We also show that we can obtain canonical grades for algebraic operations.

1 Introduction

This paper is motivated by quantitative modelling of computational effects from mathematical programming semantics. It is standard in this domain to model notions of computational effect, such as nondeterminism or manipulation of external state, by (strong) monads [11]. In many applications, however, it is useful to be able to work with quantified effects, e.g., how many outcomes a computation may have, or to what degree it may read or overwrite the state. This is relevant, for example, for program optimizations or analyses to assure that a program can run within allocated resources. Quantification of effectfulness is an old idea and goes back to type-and-effect systems [8]. Mathematically, notions of quantified effect can be modelled by graded (strong) monads [13, 10, 4].

It is natural to ask if there are systematic ways for refining a non-quantitative model of some effect into a quantitative version, i.e., for producing a graded monad from a monad. In this paper, we answer this question in the affirmative. We show how a monad on a category can be graded with any class of subfunctors (intuitively, predicates on computations) satisfying reasonable conditions, including that it forms a factorization system on some monoidal subcategory of the endofunctor category. Moreover, this grading is canonical, namely universal in a certain 2-categorical sense. We also show that algebraic operations of the given monad give rise to *flexibly graded* algebraic operations [5] of the canonically graded monad. Instead of working concretely with monads on a category, we work abstractly with monoids in a (skew) monoidal category equipped with a factorization system.

The structure of the paper is this. In Section 2, we introduce the idea of grading by subobjects for general objects and instantiate this for grading of functors. We then proceed to gradings of monoids and monads in Section 3. In Section 4, we explore the specific interesting case of grading monads canonically by subsets of their sets of shapes. In Section 5, we explain the emergence of canonical flexibly graded algebraic operations for canonical gradings of monads. One longer proof is in Appendix A.

We introduce the necessary concepts regarding the classical topics of monads, monoidal categories and factorization systems. For additional background on the more specific concepts of graded monad and skew monoidal category, which we also introduce, we refer to [4, 2] and [14, 7] as entry points.

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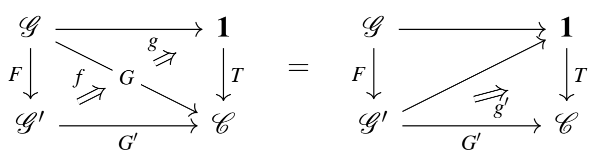
2 Grading objects and functors

As a first step towards gradings of monoids, we introduce the notion of a grading of an object of a category $\mathscr{C}$ with respect to a class of morphisms $\mathscr{M}$ in $\mathscr{C}$. We show that every object T has a canonical such grading. The case we care most about is when $\mathscr{C}$ is a category of endofunctors, so that, in the next section, where we extend these results to gradings of monoids, the monoids are exactly monads.

Definition 2.1. Let $\mathscr{G}$ be a category, whose objects e we call *grades*. A *$\mathscr{G}$-graded object* of a category $\mathscr{C}$ is a functor $G : \mathscr{G} \to \mathscr{C}$.

Let $\mathscr{M}$ be a class of morphisms of a category $\mathscr{C}$, and T be an object of $\mathscr{C}$. There is a category $\mathscr{M}/T$, which has as objects *$\mathscr{M}$-subobjects* of T, i.e., pairs (S,s) of an object S and an $\mathscr{M}$-morphism $s : S \rightarrowtail T$. Morphisms $f : (S,s) \to (S',s')$ are $\mathscr{C}$-morphisms $f : S \to S'$ such that $s = s' \circ f$. We then have a $\mathscr{M}/T$-graded object $T_{\mathscr{M}}$ of $\mathscr{C}$, defined by $T_{\mathscr{M}}(S,s) = S$. This graded object forms an $\mathscr{M}$-grading in the sense of the following definition, and is in fact the canonical $\mathscr{M}$-grading (see Theorem 2.3 below).

Definition 2.2. Let $\mathscr{M}$ be a class of morphisms of a category $\mathscr{C}$. An *$\mathscr{M}$-grading* $(\mathscr{G},G,g)$ of an object T of $\mathscr{C}$ consists of a category $\mathscr{G}$, a functor $G : \mathscr{G} \to \mathscr{C}$ (= a $\mathscr{G}$-graded object of $\mathscr{C}$), and a natural transformation typed $g_d : Gd \rightarrowtail T$ whose components are all in $\mathscr{M}$. A *morphism* $(F,f) : (\mathscr{G},G,g) \to (\mathscr{G}',G',g')$ between such gradings is a functor $F : \mathscr{G} \to \mathscr{G}'$ equipped with a natural isomorphism $f : G' \cdot F \cong G$, such that $g_d \circ f_d = g'_{Fd}$.



A *2-cell* $\beta : (F,f) \Rightarrow (F',f')$ between such morphisms is a natural transformation $\beta : F \Rightarrow F'$ such that $f' \circ (G' \cdot \beta) = f$. These form a 2-category $\mathbf{Grade}_{\mathscr{M}}T$.

We organize gradings into a 2-category so that we can prove a universal property (the following theorem) that characterizes the canonical grading. The characterization is up to equivalence; there is no reason to distinguish between isomorphic grades, therefore we can work with gradings that are equivalent to the canonical one. Working up to equivalence has the added benefit that, since $\mathscr{M}/T$ is often equivalent to a small category, we often have a canonical grading with a small set of grades.

Theorem 2.3. *Let $\mathscr{M}$ be a class of a morphisms of a category $\mathscr{C}$, and let T be an object of $\mathscr{C}$. The data $(\mathscr{M}/T, T_{\mathscr{M}}, \mathrm{snd})$, where $T_{\mathscr{M}}(S,s) = S$ and $\mathrm{snd}(S,s) = s$, make a grading of T. This grading is canonical in the sense that it is the pseudoterminal object of $\mathbf{Grade}_{\mathscr{M}}T$. Explicitly, for every other $\mathscr{M}$-grading $(\mathscr{G},G,g)$ of T:*

- *there is a morphism $(F,f) : (\mathscr{G},G,g) \to (\mathscr{M}/T, T_{\mathscr{M}}, \mathrm{snd})$ of $\mathscr{M}$-gradings;*
- *this morphism is essentially unique in the sense that there is a natural assignment of an isomorphism $(F',f') \cong (F,f)$ to every $(F',f') : (\mathscr{G},G,g) \to (\mathscr{M}/T, T_{\mathscr{M}}, \mathrm{snd})$.*

Proof. For existence, define $(F,f) : (\mathscr{G},G,g) \to (\mathscr{M}/T, T_{\mathscr{M}}, \mathrm{snd})$ by

$$Fd = (Gd, g_d) \quad Fh = Gh \qquad f_d = \mathrm{id}_{Gd}$$

For uniqueness, given (F',f'), we have 2-cells $\beta_{(F',f')} : (F',f') \Rightarrow (F,f)$ and $\beta^{-1}_{(F',f')} : (F,f) \Rightarrow (F',f')$ given by $\beta_{(F',f'),d} = f'_d$ and $\beta^{-1}_{(F',f'),d} = f'^{-1}_d$. These are clearly natural in (F',f') and inverse to each other, so we can use β as the required natural isomorphism $(F',f') \cong (F,f)$. $\square$

Remark 2.4. In this paper, we discuss the problem of constructing canonical gradings, but one can also consider the dual problem of constructing canonical *degradings*. For graded monads, this problem is discussed in [1, 9]. In the setting of this section, the initial degrading of a functor $G : \mathscr{G} \to \mathscr{C}$ would be the colimit $\operatorname{colim} G$, together with the morphisms $\mathrm{in}_e : Ge \to \operatorname{colim} G$ (when the colimit exists). The data $(\mathscr{G}, G, \mathrm{in})$ is then an $\mathscr{M}$-grading of $\operatorname{colim} G$ whenever in_e is in $\mathscr{M}$ for all e (which is the case for our examples). This grading will typically not be the canonical grading of $\operatorname{colim} G$ however. (For graded monads the situation is more complex: one does not take an ordinary colimit, but instead a colimit in a 2-category of monoidal categories, as discussed in [1].)

2.1 Canonical gradings of endofunctors on Set

We give several examples for the case where $\mathscr{C} = [\mathbf{Set}, \mathbf{Set}]$ and $\mathscr{M}$ is the class of natural transformations whose components are injective functions. In this case, every $\mathscr{M}$-subobject of an endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ is isomorphic in $\mathscr{M}/T$ to a unique $\mathscr{M}$-subobject $s : S \rightarrowtail T$ in which each injection s_X is an inclusion. In other words, s is a choice of a subset $SX \subseteq TX$ for every set X, closed under the action of T in the sense that $x \in SX$ implies $T f x \in SY$ for every $f : X \to Y$. Below we characterize $\mathscr{M}/T$ up to equivalence for various endofunctors T, using the fact that we need only consider the case where s is a family of inclusions.

Example 2.5. The category $\mathscr{M}/\mathrm{Id}$ is equivalently the poset $\{\bot \le \top\}$, with $\bot$ corresponding to the $\mathscr{M}$-subobject S given by $SX = \emptyset$ for all X, and $\top$ corresponding to $SX = X$ for all X.

Example 2.6. Consider the endofunctor $M \times T-$, where M is a set and T is an endofunctor on $\mathbf{Set}$. The category $\mathscr{M}/(M \times T-)$, is equivalent to the category $(\mathscr{M}/T)^M$, in which objects are M-indexed families $(\Sigma_z \in \mathscr{M}/T)_{z \in M}$ of $\mathscr{M}$-subobjects of T. This is the case because, from every $S \rightarrowtail M \times T-$ in which each component is an inclusion, we can construct such a family $\mathbf{\Sigma}[S]$, and this construction forms a bijection with inverse $\mathbf{S}[-]$.

$$\mathbf{\Sigma}[S]_z X = \{x \in TX \mid (z,x) \in SX\} \qquad \mathbf{S}[\Sigma]X = \{(z,x) \in M \times TX \mid x \in \Sigma_z X\}$$

In the special case $T = \mathrm{Id}$, we have $\mathscr{M}/(M \times (-)) \simeq (\mathscr{M}/\mathrm{Id})^M \simeq \{\bot \le \top\}^M \simeq (\mathscr{P}M, \subseteq)$, so the $\mathscr{M}$-subobjects of $M \times (-)$ are equivalently the subsets of M, ordered by inclusion.

Example 2.7. Consider the endofunctor $V \Rightarrow (-)$ (the underlying functor of the reader monad on $\mathbf{Set}$), where V is a fixed set, and let $\mathscr{M}$ be the class of componentwise injective natural transformations. The $\mathscr{M}$-subobjects of $V \Rightarrow (-)$, and hence the objects of the canonical $\mathscr{M}$-grading of $V \Rightarrow (-)$, are equivalently upwards-closed sets of equivalence relations on V.

To explain this in more detail, let Equiv_V be the set of equivalence relations R on V, considered as subsets $R \subseteq V \times V$. A function $f : V \to X$ *respects* $R \in \mathrm{Equiv}_V$ when $v R v'$ implies $fv = fv'$ for all $v, v' \in V$, equivalently, when f factors through the quotient $[-]_R : V \to V/R$. A set $\Sigma \subseteq \mathrm{Equiv}_V$ of equivalence relations is *upwards-closed* when $R \in \Sigma$ implies $R' \in \Sigma$ for all $R, R' \in \mathrm{Equiv}_V$ with $R \subseteq R'$. Every such Σ induces a subfunctor $\mathbf{S}[\Sigma] \rightarrowtail V \Rightarrow (-)$, defined by

$$\mathbf{S}[\Sigma]X = \{f : V \to X \mid f \text{ respects some } R \in \Sigma\}$$

To go in the other direction, consider a subfunctor $S \rightarrowtail V \Rightarrow (-)$ in which every component of the $\mathscr{M}$-morphism is an inclusion. We obtain an upwards-closed $\mathbf{\Sigma}[S] \subseteq \mathrm{Equiv}_V$:

$$\mathbf{\Sigma}[S] = \{R \in \mathrm{Equiv}_V \mid [-]_R \in S(V/R)\}$$

This is upwards-closed because if $R \subseteq R'$ then $[-]_{R'} : V \to V/R'$ factors through $[-]_R : V \to V/R$, and since S forms a functor, the family S is closed under postcomposition. These two constructions are in bijection, with $\mathbf{S}[\mathbf{\Sigma}[S]] = S$ and $\mathbf{\Sigma}[\mathbf{S}[\Sigma]] = \Sigma$. It follows that $\mathscr{M}/(V \Rightarrow (-))$ is equivalent to the poset of upwards-closed sets $\Sigma \subseteq \mathrm{Equiv}_V$, ordered by inclusion, and hence that this poset forms the canonical $\mathscr{M}$-grading of $V \Rightarrow (-)$.

Example 2.8. Consider the endofunctor $V \Rightarrow V \times (-)$ (the underlying functor of the state monad), where V is a set. Since $V \Rightarrow V \times (-) \cong (V \Rightarrow V) \times (V \Rightarrow (-))$, we can combine Examples 2.6 and 2.7 to characterize the $\mathscr{M}$-subobjects of $V \Rightarrow V \times (-)$. Every such subobject equivalently consists of an upwards-closed set $\Sigma_p \subseteq \mathrm{Equiv}_V$ for each function $p : V \to V$. These can also be seen as subsets $\Sigma \subseteq (V \Rightarrow V) \times \mathrm{Equiv}_V$ such that $\{R \mid (p,R) \in \Sigma\}$ is upwards-closed for each $p : V \to V$. Given such a Σ, the corresponding $\mathscr{M}$-subobject $\mathbf{S}[\Sigma] \rightarrowtail (V \Rightarrow V \times (-))$ is

$$\mathbf{S}[\Sigma]X = \{f : V \to V \times X \mid \exists (p,R) \in \Sigma.\ \pi_1 \circ f = p \wedge \pi_2 \circ f \text{ respects } R\}$$

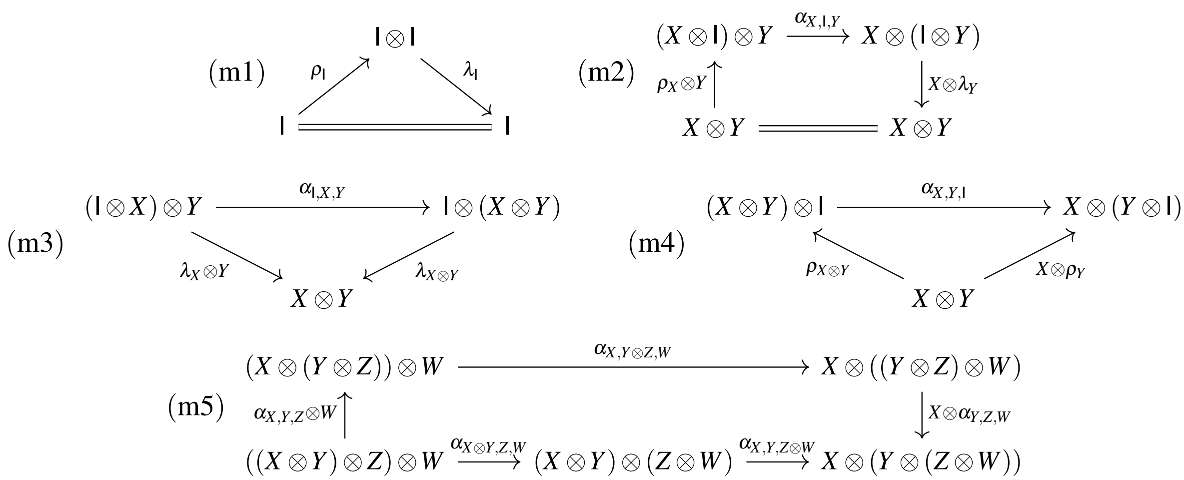
3 Grading monoids and monads

We proceed to grading monoids. To define the notion of grading of a monoid, we need an appropriate multiplication operation on the grades. The obvious idea to to ask for the grades to form a monoidal category instead of just a category, and much of the previous work on graded monads (such as [10]) does exactly this. However, in some of examples we do not get a monoidal category of grades, but only a *skew* monoidal category [14] of grades.

Definition 3.1. A *(left-)skew monoidal* category is a category $\mathscr{C}$ with a distinguished object I, a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and three natural transformations λ, ρ, α typed

$$\lambda_X : \mathsf{I} \otimes X \to X \qquad \rho_X : X \to X \otimes \mathsf{I} \qquad \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

satisfying the equations



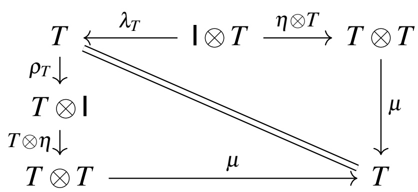
$(\mathscr{C}, \mathsf{I}, \otimes)$ is partially normal if one or several of λ, ρ or α is a natural isomorphism. In particular, it is *left-normal* if λ is an isomorphism. A monoidal category is a fully normal skew monoidal category.

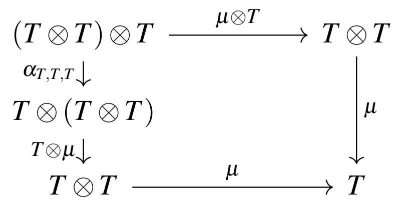
A *right-skew monoidal category* is given by $(\mathscr{C}, \mathsf{I}, \otimes, \lambda, \rho, \alpha)$ such that the data $(\mathscr{C}, \mathsf{I}, \otimes^{\mathrm{rev}}, \rho, \lambda, \alpha)$, where $X \otimes^{\mathrm{rev}} Y = Y \otimes X$, form a left-skew monoidal category.

Definition 3.2. A *monoid* in a skew monoidal category $(\mathscr{C}, \mathsf{I}, \otimes)$ is an object T of $\mathscr{C}$ equipped with morphisms

$$\eta : \mathsf{I} \to T \qquad \mu : T \otimes T \to T$$

satisfying the equations





The concept of lax monoidal functor between skew monoidal categories is defined as for monoidal categories; the same applies to the concept of monoidal transformations between lax monoidal functors.

Definition 3.3. Given a skew monoidal category $\mathbb{G} = (\mathscr{G}, I, \odot)$, a $\mathbb{G}$-*graded monoid* in a skew monoidal category $\mathbb{C} = (\mathscr{C}, \mathsf{I}, \otimes)$ is the same as a lax monoidal functor $\mathsf{G} : \mathbb{G} \to \mathbb{C}$. Explicitly, it is a functor $G : \mathscr{G} \to \mathscr{C}$ with a morphism $\eta : \mathsf{I} \to GI$ and a natural transformation typed $\mu_{d,d'} : Gd \otimes Gd' \to G(d \odot d')$ subject to equations similar to those of a monoid.

Definition 3.4. Let $\mathsf{T} = (T, \eta, \mu)$ be a monoid in a skew monoidal category $\mathbb{C} = (\mathscr{C}, \mathsf{I}, \otimes)$, and let $\mathscr{M}$ be a class of morphisms of $\mathscr{C}$. An $\mathscr{M}$-*grading* $(\mathbb{G}, \mathsf{G}, g)$ of the monoid T consists of a skew monoidal category $\mathbb{G}$, a lax monoidal functor $\mathsf{G} : \mathbb{G} \to \mathbb{C}$ (= a $\mathbb{G}$-graded monoid in $\mathbb{C}$), and a monoidal transformation typed $g_d : Gd \rightarrowtail T$, whose components are all in $\mathscr{M}$. A *morphism* $(\mathsf{F}, f) : (\mathbb{G}, \mathsf{G}, g) \to (\mathbb{G}', \mathsf{G}', g')$ between such gradings is a lax monoidal functor $\mathsf{F} : \mathbb{G} \to \mathbb{G}'$ equipped with a monoidal isomorphism $f : \mathsf{G}' \cdot \mathsf{F} \cong \mathsf{G}$, such that $g_d \circ f_d = g'_{Fd}$. A *2-cell* $\beta : (\mathsf{F}, f) \Rightarrow (\mathsf{F}', f')$ is a monoidal transformation $\beta : \mathsf{F} \Rightarrow \mathsf{F}'$ such that $f' \circ (\mathsf{G}' \cdot \beta) = f$. We write $\mathbf{Grade}_{\mathscr{M}}\mathsf{T}$ for this 2-category.

Example 3.5. The situation we are mainly interested in is when $\mathscr{C} = [\mathscr{D}, \mathscr{D}]$ is the category of endofunctors on some $\mathscr{D}$, with the identity for I and functor composition for $\otimes$. In this case, monoids in $\mathbb{C}$ are exactly monads on $\mathscr{D}$, and a lax monoidal functor $\mathsf{G} : \mathbb{G} \to \mathbb{C}$ (a $\mathbb{G}$-graded monoid in $\mathbb{C}$) is a $\mathbb{G}$-graded monad on $\mathscr{D}$, in the sense of [13, 10, 4]. Explicitly, the unit and multiplication of G have the form

$$\eta_X : X \to GIX \qquad \mu_{e,e',X} : Ge(Ge'X) \to G(e \odot e')X$$

For a concrete example, let V be a set (of states), and let T be the state monad over V:

$$TX = V \Rightarrow V \times X \qquad \eta_X\, x\, v = (v, x) \qquad \mu_X\, f\, v = g\, v' \;\text{ where } (v', g) = f\, v$$

We give a $\mathscr{M}$-grading $(\mathbb{G}, \mathsf{G}, g)$ of T, where $\mathscr{M}$ is componentwise injective natural transformations. Let $\mathscr{G}$ be the poset of subsets of $\{\mathsf{get}, \mathsf{put}\}$ ordered by inclusion, which forms a strict monoidal category with $\emptyset$ for the unit I and $e \cup e'$ for the tensor $e \odot e'$. We then define G by

$$\begin{aligned} Ge = \{ f : V \to V \times X \\ \mid\ & \mathsf{get} \notin e \Rightarrow (\pi_1 \circ f \text{ is a constant function or } \mathrm{id}_V \ \wedge\ \pi_2 \circ f \text{ is a constant function}) \\ & \wedge\ \mathsf{put} \notin e \Rightarrow \pi_1 \circ f \text{ is } \mathrm{id}_V \} \end{aligned}$$

with unit and multiplication defined as for T. This forms an $\mathscr{M}$-grading with the inclusions for g.

This grading is suitable for interpreting a Gifford-style effect system [8] for global state. A function $f : V \to V \times X$ is a computation $f \in TX$, sending an initial state to a pair of a final state and a result. A grade e gives the set of operations that a computation may use when it is executed, so GeX is the subset of TX on the computation that only use the operations in e. For example, $G\{\mathsf{get}\}X$ contains computations that may use the initial state, but do not change the state (with put).

We turn now to the *canonical* grading of a monoid T in a skew monoidal category $\mathbb{C}$. Since the category $\mathscr{M}/T$ forms the canonical grading of the *object* T, we show that (under the conditions explained below), we can make $\mathscr{M}/T$ into a skew monoidal category, using the monoid structure of T.

First consider the slice category $\mathscr{C}/T$ (where we do not restrict to $\mathscr{M}$-morphisms). This already forms a skew monoidal category, with

$$\mathsf{I} \xrightarrow{\eta} T \qquad S \otimes S' \xrightarrow{s \otimes s'} T \otimes T \xrightarrow{\mu} T$$

for the unit and tensor of (S,s) and (S',s') (see for example Kelly [6]). In general this skew monoidal structure will not restrict to $\mathscr{M}/T$, because the morphism μ is not in $\mathscr{M}$ for many of our examples. However, we can make $\mathscr{M}/T$ into a skew monoidal category by adapting the skew monoidal structure on $\mathscr{C}/T$. The idea is to just factorize the morphisms we use in the tensor and unit of $\mathscr{C}/T$ to obtain morphisms in $\mathscr{M}$. Hence we ask that $\mathscr{M}$ forms an orthogonal factorization system in the usual sense.

Definition 3.6. An (orthogonal) *factorization system* on a category $\mathscr{C}$ is a pair $(\mathscr{E}, \mathscr{M})$ of classes of morphisms of $\mathscr{C}$, such that

- both $\mathscr{E}$ and $\mathscr{M}$ contain all isomorphisms, and are closed under composition;
- $\mathscr{E}$-morphisms are *orthogonal* to $\mathscr{M}$-morphisms: for every commuting square in $\mathscr{C}$ as on the left below, with $e \in \mathscr{E}$ and $m \in \mathscr{M}$, there is a unique d making the diagram on the right below commute.

$$\begin{array}{ccc} X & \overset{e}{\twoheadrightarrow} & Y \\ f\downarrow & & \downarrow g \\ X' & \underset{m}{\rightarrowtail} & Y' \end{array} \qquad\qquad \begin{array}{ccc} X & \overset{e}{\twoheadrightarrow} & Y \\ f\downarrow & \swarrow d & \downarrow g \\ X' & \underset{m}{\rightarrowtail} & Y' \end{array}$$

- every morphism $f : X \to Y$ in $\mathscr{C}$ has an $(\mathscr{E}, \mathscr{M})$-factorization: there exist an object S, an $\mathscr{E}$-morphism $e : X \twoheadrightarrow S$, and an $\mathscr{M}$-morphism $m : S \rightarrowtail Y$ such that the diagram below commutes.

$$X \xrightarrow{f} Y \qquad = \qquad X \overset{e}{\twoheadrightarrow} S \overset{m}{\rightarrowtail} Y$$

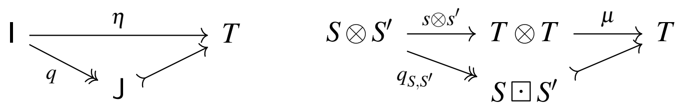
Example 3.7. Writing **Mor** for the class of all morphisms and **Iso** for the class of isomorphisms, every category has $(\mathbf{Mor}, \mathbf{Iso})$ and $(\mathbf{Iso}, \mathbf{Mor})$ as factorization systems. On **Set**, the classes of surjective and of injective functions form a factorization system $(\mathbf{Surj}, \mathbf{Inj})$. To factorize a function $f : X \to Y$ in this case, we let $S = \{f\,x \mid x \in X\}$ be the image of f, and define $X \overset{e}{\twoheadrightarrow} S \overset{m}{\rightarrowtail} Y$ by $e\,x = f\,x$ and $m\,y = y$. On the category **Poset** of partially ordered sets and monotone functions, we have a factorization system $(\mathbf{Surj}, \mathbf{Full})$, where **Surj** is the class of surjective monotone functions and **Full** is the class of full functions, i.e. monotone functions $m : S \to Y$ such that $m\,x \leq m\,y$ implies $x \leq y$. Factorizations are given as in **Set**, with the order on S inherited from the order on Y.

We are primarily interested in canonically grading monads, which are monoids in the endofunctor category $\mathscr{C} = [\mathscr{D}, \mathscr{D}]$, with functor composition as the tensor. We therefore want a factorization system on $[\mathscr{D}, \mathscr{D}]$. We give the most standard option for this as the following example, but there are others we are interested in (see Lemma 4.2 below). In models of computational effects we in fact usually want a *strong* monad. If $\mathscr{D}$ is monoidal, then a *strong endofunctor* on $\mathscr{D}$ is a functor $F : \mathscr{D} \to \mathscr{D}$ equipped with a *strength*, i.e. a natural transformation $\mathrm{str}_{\Gamma,X} : \Gamma \otimes_{\mathscr{D}} FX \to F(\Gamma \otimes_{\mathscr{D}} X)$ satisfying two laws for compatibility with the left unitor and associator of $\mathscr{D}$. These form a monoidal category $[\mathscr{D}, \mathscr{D}]_s$, in which morphisms are strength-preserving natural transformations and the tensor is composition. Strong monads are monoids in $[\mathscr{D}, \mathscr{D}]_s$. Below we consider non-strong monads for simplicity, but we can also apply our results to strong monads using a factorization system on $[\mathscr{D}, \mathscr{D}]_s$.

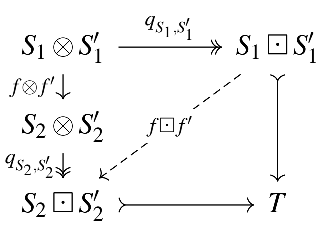
Example 3.8. If $(\mathscr{E},\mathscr{M})$ is a factorization system on a category $\mathscr{D}$, then the endofunctor category $[\mathscr{D},\mathscr{D}]$ has a factorization system (componentwise-$\mathscr{E}$, componentwise-$\mathscr{M}$). Factorizations $F \twoheadrightarrow S \rightarrowtail G$ of natural transformations are componentwise. If $\mathscr{D}$ is monoidal and $\mathscr{E}$ is closed under $\Gamma \otimes_{\mathscr{D}} (-)$ for all Γ then (componentwise-$\mathscr{E}$, componentwise-$\mathscr{M}$) is a factorization system on $[\mathscr{D},\mathscr{D}]_s$. Morphisms are again factorized componentwise; for the construction of the strength for S see [3, Section 2.2].

Forming a factorization system $(\mathscr{E},\mathscr{M})$ is merely a property of a class $\mathscr{M}$ of morphisms, because $\mathscr{E}$ is necessarily the class of all morphisms e that are orthogonal to all $\mathscr{M}$-morphisms. Factorizations of morphisms are unique up to unique isomorphism.

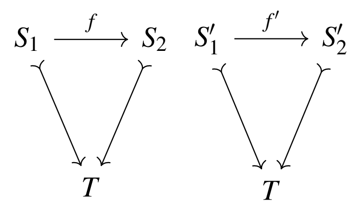
If $\mathscr{M}$ forms a factorization system $(\mathscr{E},\mathscr{M})$, then for a given monoid T we construct a unit J and a tensor $\boxdot$ for the category $\mathscr{M}/T$ by factorizing morphisms as follows:



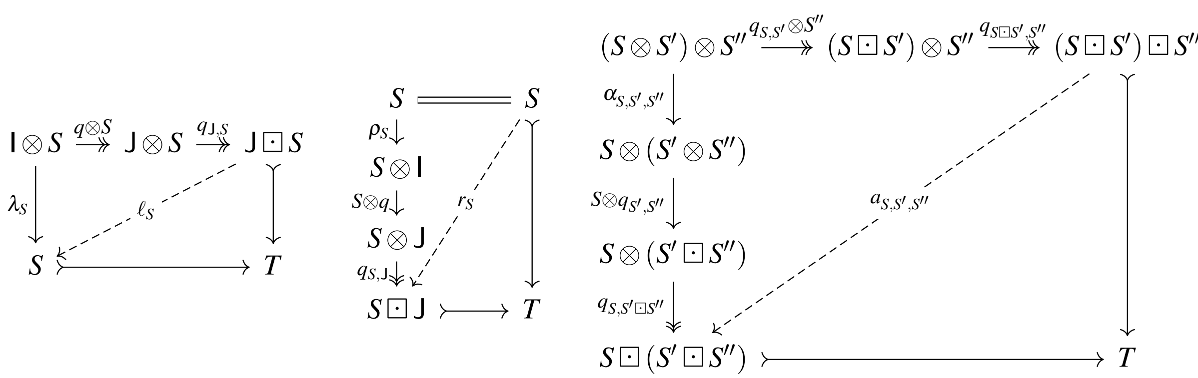
To construct the required structural morphisms, we make the additional assumption that $\mathscr{E}$ is closed under $(-) \otimes S$ for every $S \rightarrowtail T$. Under this assumption, the following squares, which all commute, have a unique diagonal, and these diagonals are the required structural morphisms.



where



are morphisms in $\mathscr{M}/T$



If λ is a natural isomorphism, then so is ℓ. This does not apply to ρ and α unless $\mathscr{E}$ is also closed under $S \otimes (-)$ for all $S \rightarrowtail T$.

Theorem 3.9. *Let T be a monoid in a skew monoidal category $\mathbb{C} = (\mathscr{C},\mathsf{I},\otimes)$, and let $(\mathscr{E},\mathscr{M})$ be a factorization system on $\mathscr{C}$. If $\mathscr{E}$ is closed under $(-) \otimes S$ for every $\mathscr{M}$-subobject $S \rightarrowtail T$, then $\mathscr{M}/\mathsf{T} = (\mathscr{M}/T,\mathsf{J},\boxdot)$ is a skew monoidal category. If $\mathbb{C}$ is left-normal, then so is $\mathscr{M}/\mathsf{T}$. If $\mathscr{E}$ is also closed under $S \otimes (-)$ for every $S \rightarrowtail T$ and $\mathbb{C}$ is monoidal, then $\mathscr{M}/\mathsf{T}$ is monoidal.*

Proof. See Appendix A. □

Remark 3.10. Closure of $\mathscr{E}$ under $T \otimes (-)$ does not in general imply closure of $\mathscr{E}$ under $S \otimes (-)$ for $S \rightarrowtail T$. Consider the factorization system $(\mathscr{E}, \mathscr{M}) = (\text{componentwise surjective}, \text{componentwise full})$ on $[\mathbf{Poset}, \mathbf{Poset}]$, with composition as $\otimes$. For an endofunctor $F : \mathbf{Poset} \to \mathbf{Poset}$, closure of $\mathscr{E}$ under $F \otimes (-)$ amounts to closure of **Surj** under F. The class **Surj** is closed under $V \Rightarrow (-)$ exactly when V is discrete. Hence, while this property holds for $\{0,1\} \Rightarrow (-)$, it does not hold for $\{0 \leq 1\} \Rightarrow (-) \rightarrowtail \{0,1\} \Rightarrow (-)$.

There are cases in which $\mathscr{E}$ is closed under $S \otimes (-)$ for all $S \rightarrowtail T$ even if $\mathscr{E}$ is not closed under $F \otimes (-)$ for general F: for example every $\mathscr{M}$-subobject of the endofunctor $M \times (-)$ on **Poset** has the form $S \times (-)$ for some $S \rightarrowtail M$, and functors of the form $S \times (-)$ send surjections to surjections.

Our task is now to show that the skew monoidal category $\mathscr{M}/\mathsf{T} = (\mathscr{M}/T, \mathsf{J}, \boxdot)$ forms the canonical grading $(\mathscr{M}/\mathsf{T}, \mathsf{T}_{\mathscr{M}}, \mathrm{snd})$ of the monoid T when $\mathscr{E}$ is closed under $S \otimes (-)$ for each $S \rightarrowtail T$. The lax monoidal functor $\mathsf{T}_{\mathscr{M}} : \mathscr{M}/\mathsf{T} \to \mathbb{C}$ is given on objects by

$$T_{\mathscr{M}}(S, s : S \rightarrowtail T) = S$$

and has as unit and multiplication the $\mathscr{E}$-morphisms from the construction of J and $\boxdot$:

$$\mathsf{I} \xrightarrow{q} T_{\mathscr{M}}\mathsf{J} \qquad\qquad T_{\mathscr{M}}(S,s) \otimes T_{\mathscr{M}}(S',s') \xrightarrow{q_{S,S'}} T_{\mathscr{M}}((S,s) \boxdot (S',s'))$$

That this is lax monoidal is immediate from the definition of the structural morphisms of $\mathscr{M}/\mathsf{T}$. Finally, the monoidal transformation $\mathrm{snd} : \mathsf{T}_{\mathscr{M}} \Rightarrow \mathsf{T}$ is given by $\mathrm{snd}_{(S,s)} = s$. Monoidality of snd is immediate from the definitions of J and $\boxdot$. Hence $(\mathscr{M}/\mathsf{T}, \mathsf{T}_{\mathscr{M}}, \mathrm{snd})$ is a grading of T. Canonicity is the following theorem.

Theorem 3.11. *Let T be a monoid in a skew monoidal category $\mathbb{C} = (\mathscr{C}, \mathsf{I}, \otimes)$, and let $(\mathscr{E}, \mathscr{M})$ be a factorization system on $\mathscr{C}$ such that $\mathscr{E}$ is closed under $(-) \otimes S$ for each $\mathscr{M}$-subobject S of T. The grading $(\mathscr{M}/\mathsf{T}, \mathsf{T}_{\mathscr{M}}, \mathrm{snd})$ is canonical in the sense that it is the pseudoterminal object of $\mathbf{Grade}_{\mathscr{M}}\mathsf{T}$. Explicitly, for every $\mathscr{M}$-grading $(\mathbb{G}, \mathsf{G}, g)$ of the monoid T:*

- *there is a morphism $(\mathsf{F}, f) : (\mathbb{G}, \mathsf{G}, g) \to (\mathscr{M}/\mathsf{T}, \mathsf{T}_{\mathscr{M}}, \mathrm{snd})$, of $\mathscr{M}$-gradings of T;*
- *this morphism is essentially unique in the sense that there is a natural assignment of an isomorphism $(\mathsf{F}', f') \cong (F, f)$ to every $(\mathsf{F}', f') : (\mathbb{G}, \mathsf{G}, g) \to (\mathscr{M}/\mathsf{T}, \mathsf{T}_{\mathscr{M}}, \mathrm{snd})$.*

Proof. We have done most of the proof already as Theorem 2.3; we fill in the remaining parts. Recall from there that we define

$$Fd = (Gd, g_d) \quad Fh = Gh \qquad f_d = \mathrm{id}_{Gd}$$

We make F into a lax monoidal functor by using the unique diagonals of the following squares as the unit and multiplication. The squares commute because G is lax monoidal.

$$\begin{array}{ccc} \mathsf{I} & \xrightarrow{q} & \mathsf{J} \\ {\scriptstyle\eta}\downarrow & \swarrow{\scriptstyle\eta} & \downarrow \\ GI & \xrightarrow[g_I]{} & T \end{array} \qquad\qquad \begin{array}{ccc} Gd \otimes Gd' & \xrightarrow{q_{Gd,Gd'}} & Gd \boxdot Gd' \\ {\scriptstyle\mu_{d,d'}}\downarrow & \swarrow{\scriptstyle\mu_{d,d'}} & \downarrow \\ G(d \odot d') & \xrightarrow[g_{d \odot d'}]{} & T \end{array}$$

This definition immediately implies that f is monoidal, and hence that (F, f) is a morphism of $\mathscr{M}$-gradings of T. Finally, given (F', f'), we show that $\beta_d = f'_d$ defines an isomorphism $\beta : (\mathsf{F}', f') \cong (\mathsf{F}, f)$. For this it remains to show that β is monoidal (it follows automatically that the inverse β^{-1} is monoidal). For compatibility with the multiplications, this amounts to showing that the square on the left below commutes. For this it is enough to show both paths in that square provide the unique diagonal of the square on the right, and this follows from the fact that f' is monoidal.

$$\begin{array}{ccc} F'd \boxdot F'd' & \xrightarrow{f'_d \boxdot f'_{d'}} & Gd \boxdot Gd' \\ \downarrow{\scriptstyle \mu_{d,d'}} & & \downarrow{\scriptstyle \mu_{d,d'}} \\ F'(d \odot d') & \xrightarrow[f'_{d \odot d'}]{} & G(d \odot d') \end{array} \qquad \begin{array}{ccc} F'd \otimes F'd' & \xrightarrow{q_{F'd,F'd'}} & F'd \boxdot F'd' \\ \downarrow{\scriptstyle f'_d \otimes f'_{d'}} & & \\ Gd \otimes Gd' & \swarrow & \downarrow \\ \downarrow{\scriptstyle \mu_{d,d'}} & & \\ G(d \odot d') & \xrightarrow[g_{d \boxdot d'}]{} & T \end{array}$$

Compatibility with the units is similar. □

Example 3.12. Let $(M, \varepsilon, \cdot)$ be a monoid in the cartesian monoidal category **Set**, and let T be the corresponding writer monad, which has endofunctor $TX = M \times X$, unit $\eta_X x = (\varepsilon, x)$, and multiplication $\mu_X(z, (z', x)) = (z \cdot z', x)$. Then T is a monoid in the monoidal category of endofunctors on **Set**, with functor composition. Consider the factorization system $(\mathscr{E}, \mathscr{M})$ on $[\mathbf{Set}, \mathbf{Set}]$ in which $\mathscr{E}$ (respectively $\mathscr{M}$) is componentwise surjective (resp. injective) natural transformations. The class $\mathscr{E}$ is closed under functor composition on both sides, so $\mathscr{M}/\mathsf{T}$ is monoidal and provides the canonical grading of T. We show in Example 2.6 that $\mathscr{M}$-subobjects of T are equivalently subsets $\Sigma \subseteq M$. Under this equivalence, the monoidal structure on $\mathscr{M}/\mathsf{T}$ is given by $\mathsf{J} = \{\varepsilon\}$ and $\Sigma \boxdot \Sigma' = \{z \cdot z' \mid z \in \Sigma, z' \in \Sigma'\}$. The graded monad $\mathsf{T}_{\mathscr{M}}$ is given by

$$T_{\mathscr{M}}\Sigma = \Sigma \times X \qquad \eta_X x = (\varepsilon, x) \qquad \mu_{\Sigma,\Sigma',X}(z, (z', x)) = (z \cdot z', x)$$

Example 3.13. We show in Example 2.8 that, when $\mathscr{M}$ is componentwise injective natural transformations and $T = V \Rightarrow V \times (-)$, the objects of $\mathscr{M}/T$ are equivalently subsets $\Sigma \subseteq (V \Rightarrow V) \times \mathrm{Equiv}_V$ satisfying a closure condition. When T is the state monad, these form a monoidal category $\mathscr{M}/\mathsf{T}$, and the graded monad $\mathsf{T}_{\mathscr{M}}$ has underlying functor

$$T_{\mathscr{M}}\Sigma X = \{f : V \to V \times X \mid \exists (p, R) \in \Sigma.\, p = \pi_1 \circ f \wedge (\pi_2 \circ f) \text{ respects } R\}$$

Example 3.5 provides another grading of T, in which the grades are subsets of $\{\mathsf{get}, \mathsf{put}\}$. By Theorem 3.11, we obtain a morphism (F, f) of gradings. Under the characterization of grades as subsets Σ, the underlying functor F sends $e \subseteq \{\mathsf{get}, \mathsf{put}\}$ to $Fe \subseteq (V \Rightarrow V) \times \mathrm{Equiv}_V$ as follows:

$$F\emptyset = \{(\mathrm{id}_V, V \times V)\} \qquad F\{\mathsf{get}, \mathsf{put}\} = (V \Rightarrow V) \times \mathrm{Equiv}_V$$
$$F\{\mathsf{get}\} = \{(\mathrm{id}_V, R) \mid R \in \mathrm{Equiv}_V\} \qquad F\{\mathsf{put}\} = \{(p, V \times V) \mid p \text{ is a constant function or } \mathrm{id}_V\}$$

4 Canonical grading by sets of shapes

When assigning grades to computations $t \in TX$, where T is a monad on **Set**, we are often interested only in the *shape* of the computation. A *shape* is an element of $T1$, where 1 is the one-element set; and the shape of the computation $t \in TX$ is $T!t \in T1$, where ! is the unique function $X \to 1$. A grade in this case is a subset $e \subseteq T1$ of the set of shapes, and a computation has grade e when its shape $T!t$ is in e.

More generally, if T is a monad on a category $\mathscr{D}$ with a terminal object 1, then the object of shapes is $T1$. Given a class $\mathscr{M}$ of morphisms of $\mathscr{D}$, we can consider grading by $\mathscr{M}$-subobjects of $T1$. We show in this section that these grades can be considered canonical, using a suitable class $\mathscr{M}'$ of morphisms of $[\mathscr{D}, \mathscr{D}]$.

Definition 4.1. Let $\mathscr{A}$ be a category. A natural transformation $f : F \Rightarrow G : \mathscr{A} \to \mathscr{D}$ is *cartesian* if all of its naturality squares are pullbacks. If $\mathscr{A}$ has pullbacks, then a functor $F : \mathscr{A} \to \mathscr{D}$ is *cartesian* when it preserves pullbacks.

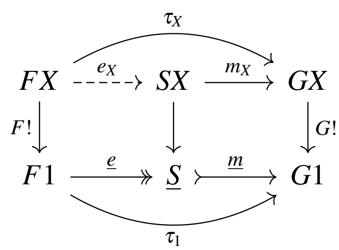
Lemma 4.2. *Let $(\mathscr{E}, \mathscr{M})$ be a factorization system on a category $\mathscr{D}$ with pullbacks, and let $\mathscr{A}$ be a category with a terminal object. Then we have a factorization system $(\mathscr{E}', \mathscr{M}')$ on $[\mathscr{A}, \mathscr{D}]$ as follows:*

$$\mathscr{E}' = \text{natural transformations } e \text{ such that } e_1 \in \mathscr{E}$$
$$\mathscr{M}' = \text{cartesian natural transformations } m \text{ such that } m_1 \in \mathscr{M}$$

For every functor $G : \mathscr{A} \to \mathscr{D}$, there is an equivalence of categories $\mathscr{M}'/G \simeq \mathscr{M}/G1$.

Before giving the proof, we note that Kelly [6] considers $(\mathscr{E}', \mathscr{M}')$ in the case $(\mathscr{E}, \mathscr{M}) = (\mathbf{Iso}, \mathbf{Mor})$.

Proof. That $\mathscr{E}'$ and $\mathscr{M}'$ are closed under isomorphisms and composition is straightforward. To factorize a natural transformation $\tau : F \Rightarrow G$, we first factorize τ_1 using $(\mathscr{E}, \mathscr{M})$, as on the bottom of the following diagram.



We then factorize any component τ_X as on the top, by taking as (SX, m_X) the pullback of $(\underline{S}, \underline{m})$ along $G!$, and taking as e_X the unique map from FX to this pullback. The objects SX form a functor using unique maps into pullbacks, and the morphisms e_X and m_X are natural transformations. When $X = 1$ we have $e_X \in \mathscr{E}$ and $m_X \in \mathscr{M}$ because the vertical morphisms in the diagram are all isomorphisms. Hence we have the required factorization of τ into $e \in \mathscr{E}'$ and $m \in \mathscr{M}'$. For orthogonality, unique diagonal fill-ins are constructed as unique maps to pullbacks.

The required equivalence of categories exists because every $\mathscr{M}'$-subobject $m : S \rightarrowtail G$ is determined up to isomorphism by the component m_1. The latter is the corresponding object of $\mathscr{M}/G1$. □

Lemma 4.2 provides a construction of a factorization system $(\mathscr{E}', \mathscr{M}')$ in particular on endofunctor categories $[\mathscr{D}, \mathscr{D}]$ when $\mathscr{D}$ has pullbacks and a terminal object. In this case the $\mathscr{M}'$-subobjects of T are $\mathscr{M}$-subobjects of $T1$. However, $\mathscr{E}'$ is often closed under neither $(-) \cdot S$ nor $S \cdot (-)$ for $S \rightarrowtail T$, and $\mathscr{M}'/T$ is neither left-skew nor right-skew monoidal for a monad T. In the following example, left-skew monoidality fails, but we do get right-skew monoidality.

Example 4.3. Consider the factorization system $(\mathscr{E}, \mathscr{M}) = (\mathbf{Surj}, \mathbf{Inj})$ on $\mathscr{A} = \mathscr{D} = \mathbf{Set}$. Surjections in $\mathbf{Set}$ are preserved by any functor S, therefore the class $\mathscr{E}'$ of the factorization system $(\mathscr{E}', \mathscr{M}')$ on $[\mathbf{Set}, \mathbf{Set}]$ is closed under $S \cdot (-)$ for any S (as $(S \cdot e)_1 = Se_1$). Given a set monad T, the category $\mathscr{M}'/T$ obtains a right-skew monoidal structure by the "reversal" of Theorem 3.9.

Let T be the state monad for a set of states V:

$$T = V \Rightarrow V \times (-) \qquad \eta_X\, x\, v = (v, x) \qquad \mu_X\, f\, v = g\, v' \;\text{ where } (v', g) = f\, v$$

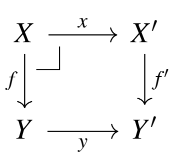
Since $T1 \cong V \Rightarrow V$, we have $\mathscr{M}'/T \simeq \mathbf{Inj}/(V \Rightarrow V)$, so that the canonical grades are equivalently subsets of $\Sigma \subseteq V \Rightarrow V$, ordered by inclusion. A subset Σ corresponds to the $\mathscr{M}'$-subobject $\mathbf{S}[\Sigma] \rightarrowtail T$ given by

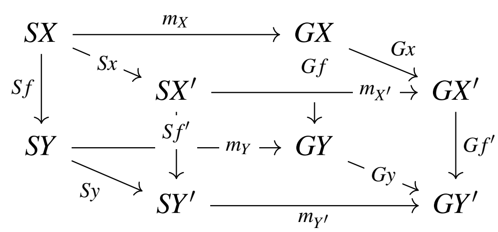
$\mathbf{S}[\Sigma]X = \{f : V \to V \times X \mid \pi_1 \circ f \in \Sigma\}$. Given such a subset, define $\mathrm{Cl}(\Sigma') = \{f : V \to V \times X \mid \forall v. \exists g \in \Sigma'. \pi_1(f\,v) = g\,v\}$. The right-skew monoidal category of canonical grades Σ has unit $\mathsf{J} = \{\mathrm{id}_V\}$ and tensor $\Sigma \boxdot \Sigma' = \Sigma \,\hat{\circ}\, \mathrm{Cl}(\Sigma')$, where $\Sigma \,\hat{\circ}\, \Sigma' = \{f' \circ f \mid f \in \Sigma, f' \in \Sigma'\}$. There is no left unitor for a left-skew monoidal structure, because $\mathsf{J} \boxdot \Sigma' = \mathrm{Cl}(\Sigma')$ is not in general equal to Σ'.

The failure of left-skew monoidality in this example can be traced back to the failure of $\mathscr{E}'$ to be closed under $(-) \cdot S$ for endofunctors $S \rightarrowtail T$. We had no such problem for the componentwise lifting of $(\mathscr{E}, \mathscr{M})$. When $(\mathscr{E}, \mathscr{M})$ is *stable* (Definition 4.6 below) and one restricts $[\mathscr{A}, \mathscr{D}]$ to cartesian natural transformations (and optionally further also to cartesian functors), then the componentwise lifting actually coincides with $(\mathscr{E}', \mathscr{M}')$, as we show in the next few lemmata. This then provides sufficient conditions for $\mathscr{E}'$ to be closed under $(-) \cdot S$. We give an example in which these conditions are satisfied in Example 4.10 below, where we actually obtain a monoidal structure on the category of grades.

Lemma 4.4. *If $\mathscr{A}$ has pullbacks, $m : S \Rightarrow G : \mathscr{A} \to \mathscr{D}$ is a cartesian natural transformation, and G is cartesian, then S is also cartesian.*

Proof. Every pullback square in $\mathscr{A}$, as on the left below, induces a cube in $\mathscr{D}$, as on the right below. Four of the faces of this cube are pullbacks because m is cartesian, and the face on the right is a pullback because G is cartesian. It follows that the left face is also a pullback.





□

Lemma 4.5. *If $\mathscr{A}$ has pullbacks, then any factorization system on $[\mathscr{A}, \mathscr{D}]_{\mathrm{cartnt}}$ (all functors, but only cartesian natural transformations) restricts to a factorization system on $[\mathscr{A}, \mathscr{D}]_{\mathrm{cart}}$ (cartesian functors and cartesian natural transformations).*

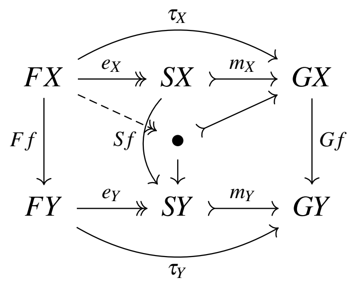
Proof. It suffices to show that $[\mathscr{A}, \mathscr{D}]_{\mathrm{cart}}$ is closed under factorizations of cartesian natural transformations. This a consequence of the previous lemma: if $\tau : F \Rightarrow G$ is a morphism in $[\mathscr{A}, \mathscr{D}]_{\mathrm{cart}}$ that factorizes as (S, e, m), then S is cartesian because G and m are. □

Definition 4.6. If $\mathscr{D}$ has pullbacks, then a factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathscr{D}$ is *stable* when $\mathscr{E}$ is closed under pullbacks along arbitrary morphisms.

(The analogous property for $\mathscr{M}$ is true in every factorization system.)

Lemma 4.7. *Assume that $\mathscr{D}$ has pullbacks, and let $(\mathscr{E}, \mathscr{M})$ be a stable factorization system on $\mathscr{D}$. The componentwise lifting of $(\mathscr{E}, \mathscr{M})$ to $[\mathscr{A}, \mathscr{D}]$ restricts to a factorization system on $[\mathscr{A}, \mathscr{D}]_{\mathrm{cartnt}}$.*

Proof. We need to check that, if a cartesian natural transformation $\tau : F \Rightarrow G$ factorizes as (S, e, m) using $(\mathscr{E}, \mathscr{M})$ componentwise, then e and m are cartesian natural transformations. Given any $f : X \to Y$, we can consider the naturality square of τ for f, which is by assumption cartesian. It breaks into naturality squares of e and m for f. We can then pull back (SY, m_Y) along Gf and be certain that the resulting morphism is in $\mathscr{M}$. The unique morphism from FX to the pullback vertex $\bullet$ is a pullback of (FY, e_Y) along Sf by the pullback lemma, and is therefore in $\mathscr{E}$ by stability. We therefore have two factorizations of τ_X: one through SX and one through $\bullet$. Factorizations are unique up to isomorphism, and hence the naturality squares of both e and m are pullbacks.



We also need to check that the diagonal fill-ins of cartesian squares built using $(\mathscr{E},\mathscr{M})$ componentwise are cartesian. This holds because the pullback lemma provides a two-out-of-three property for cartesian natural transformations: if $m \circ d$ and m are cartesian, then d is also cartesian. □

Lemma 4.7 enables us to restrict the componentwise lifting of a factorization system to $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$. The following lemma enables us to restrict the factorization system $(\mathscr{E}',\mathscr{M}')$ defined at the beginning of this section.

Lemma 4.8. *Let $(\mathscr{E},\mathscr{M})$ be any factorization system on $[\mathscr{A},\mathscr{D}]$. If all natural transformations in $\mathscr{M}$ are cartesian, then $(\mathscr{E},\mathscr{M})$ restricts to a factorization system on $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$.*

Proof. If $\tau = m \circ e$ and τ and m are cartesian, then e is cartesian by the pullback lemma. □

Lemmata 4.7 and 4.8 provide two constructions of a factorization system on $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$: the componentwise lifting of a factorization system on $\mathscr{D}$, and the factorization system $(\mathscr{E}',\mathscr{M}')$ of Lemma 4.2. We now show that the two coincide.

Proposition 4.9. *Let $(\mathscr{E},\mathscr{M})$ be a stable factorization system on a category $\mathscr{D}$ with pullbacks, and let $\mathscr{A}$ be a category with a terminal object. The factorization system $(\mathscr{E}',\mathscr{M}')$ on $[\mathscr{A},\mathscr{D}]$ from Lemma 4.2 and the componentwise lifting of $(\mathscr{E},\mathscr{M})$ to $[\mathscr{A},\mathscr{D}]$ both restrict to the same factorization system on $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$.*

Proof. By stability of $(\mathscr{E},\mathscr{M})$, for each cartesian natural transformation e, having $e_1 \in \mathscr{E}$ is equivalent to having $e_X \in \mathscr{E}$ for all X. Hence when restricted to $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$, the $\mathscr{E}'$-morphisms are exactly the $[\mathscr{A},\mathscr{D}]_{\mathrm{cartnt}}$-morphisms whose components are in $\mathscr{E}$, and similarly for $\mathscr{M}'$. □

When $\mathscr{A}$ has pullbacks it follows, using Lemma 4.5, that the two factorization systems also restrict to the same factorization system on $[\mathscr{A},\mathscr{D}]_{\mathrm{cart}}$. When $\mathscr{A} = \mathscr{D}$, the latter forms a monoidal category with functor composition as tensor, and $\mathscr{E}'$ is closed under $(-)\cdot S$ for every cartesian endofunctor S. Hence we can construct canonical gradings of cartesian monads (monoids in $[\mathscr{D},\mathscr{D}]_{\mathrm{cart}}$) by Theorem 3.11. The grades of the canonical grading of a cartesian monad T are equivalently $\mathscr{M}$-subobjects of $T1$, by Lemma 4.2.

Example 4.10. Let us return to the factorization system $(\mathscr{E},\mathscr{M}) = (\mathbf{Surj},\mathbf{Inj})$ on $\mathscr{A} = \mathscr{D} = \mathbf{Set}$, which is stable. Let $(\mathscr{E}',\mathscr{M}')$ be the factorization system on $[\mathbf{Set},\mathbf{Set}]_{\mathrm{cart}}$ just discussed. Then $\mathscr{E}'$ is closed both under $(-)\cdot S$ and under $S\cdot(-)$ for any cartesian set functor S. Hence $\mathscr{M}'/T$ acquires a monoidal structure for any cartesian set monad T.

Let T be the list monad on **Set**, so TX is the set of lists over X, the unit is $\eta_X\, x = [x]$, and the multiplication is $\mu_X\,[xs_1,\ldots,xs_n] = xs_1 \mathbin{+\!\!+} \cdots \mathbin{+\!\!+} xs_n$, where $(\mathbin{+\!\!+})$ is concatenation of lists. This monad is cartesian. There is an isomorphism $T1 \cong \mathbb{N}$, so shapes are equivalently natural numbers (corresponding to the length of the list). Then $\mathscr{M}'$-subobjects of T are equivalently subsets $\Sigma \subseteq \mathbb{N}$. By the above, these form the canonical $\mathscr{M}'$-grading of T. The monoidal structure on these subsets is given by

$$\mathsf{J} = \{1\} \qquad \Sigma \boxdot \Sigma' = \{\textstyle\sum_{i=1}^{n} m_i \mid n \in \Sigma, m_1,\ldots,m_n \in \Sigma'\}$$

The graded monad $\mathsf{T}_{\mathscr{M}'}$ is given on objects by $\mathsf{T}_{\mathscr{M}'}\Sigma X = \{xs \mid |xs| \in \Sigma\}$, where $|xs|$ is the length of xs.

5 Algebraic operations

In models of computational effects, we usually do not just want a (strong) monad T; we also want to equip T with a collection of *algebraic operations* in the sense of Plotkin and Power [12]. The latter provide interpretations of the constructs that cause the effects. When modelling computations using a graded monad, we similarly want algebraic operations for the graded monad; such a notion of algebraic operation was introduced in [5]. In this section, we therefore investigate the problem of constructing algebraic operations for the graded monad $\mathsf{T}_{\mathscr{M}}$, given algebraic operations for the monad T.

Throughout this section, we assume a monoidal category $\mathbb{C} = (\mathscr{C}, \mathsf{I}, \otimes)$ that has finite products, for example, endofunctors on a category with finite products. When we write T^n below, we mean the product of n-many copies of T. We work only with normal (i.e., non-skew) monoidal categories in this section. The notion of algebraic operation for a graded monoid (e.g., a graded monad) that we use below works for monoidal categories, but the appropriate notion for skew monoidal categories would be more complicated. (It would use a list of grades e_i instead of a single grade e in the definition below.) Hence when we consider the canonical gradings below, we work under the assumption that they form a monoidal category (for example, when $\mathscr{E}$ is closed under $\otimes$ in both arguments).

The following definition generalizes the notion of algebraic operation for a monad to monoids.

Definition 5.1. Let $\mathsf{T} = (T, \eta, \mu)$ be a monoid in $\mathbb{C}$. An n-ary *algebraic operation* for T, where n is a natural number, is a morphism $\phi : T^n \to T$ such that

$$\begin{array}{ccccc} T^n \otimes T & \xrightarrow{\langle \pi_i \otimes T\rangle_i} & (T \otimes T)^n & \xrightarrow{\mu^n} & T^n \\ {\scriptstyle \phi\otimes T}\downarrow & & & & \downarrow{\scriptstyle \phi} \\ T \otimes T & & \xrightarrow{\qquad\mu\qquad} & & T \end{array}$$

Definition 5.2. Let $\mathsf{G} = (G, \eta, \mu) : \mathbb{G} \to \mathbb{C}$ be a $\mathbb{G}$-graded monoid in $\mathbb{C}$. A $(d_1,\ldots,d_n; d')$-ary algebraic operation for G, where $d_1,\ldots,d_n,d' \in \mathbb{G}$, is a natural transformation $\psi_e : \prod_i G(d_i \odot e) \Rightarrow G(d' \odot e)$ such that, for all $e, e' \in \mathscr{G}$,

$$\begin{array}{ccccccc} (\prod_i G(d_i \odot e)) \otimes Ge' & \xrightarrow{\langle \pi_i \otimes Ge'\rangle_i} & \prod_i (G(d_i \odot e) \otimes Ge') & \xrightarrow{\prod_i \mu_{d_i \odot e, e'}} & \prod_i G((d_i \odot e) \odot e') & \xrightarrow{G\alpha} & \prod_i G(d_i \odot (e \odot e')) \\ {\scriptstyle \psi_e \otimes Ge'}\downarrow & & & & & & \downarrow{\scriptstyle \psi_{e\odot e'}} \\ G(d' \odot e) \otimes Ge' & & \xrightarrow{\qquad\mu_{d'\odot e, e'}\qquad} & & G((d' \odot e) \odot e') & \xrightarrow{G\alpha} & G(d' \odot (e \odot e')) \end{array}$$

Example 5.3. Let $\mathbb{C}$ be the cartesian monoidal category **Set**. Then an n-ary algebraic operation for a monoid T is a function $\phi : T^n \to T$ such that the multiplication of the monoid distributes over ϕ from the right. For example, if T is natural numbers with ordinary multiplication, then $\phi(x_1,\ldots,x_n) = x_1 + \cdots + x_n$ is an n-ary algebraic operation.

Definition 5.4. Let $(\mathbb{G}, \mathsf{G}, g)$ be an $\mathscr{M}$-grading of a monoid T, where $\mathscr{M}$ is a class of morphisms in $\mathscr{C}$. We say that a $(d_1, \ldots, d_n; d')$-ary algebraic operation ψ for G is a *grading* of an n-ary algebraic operation ϕ for T when the following diagram commutes for all $e \in \mathbb{G}$.

$$\begin{array}{ccc}
\prod_i G(d_i \odot e) & \xrightarrow{\psi_e} & G(d' \odot e) \\
{\scriptstyle \prod_i g_{d_i \odot e}} \downarrow & & \downarrow {\scriptstyle g_{d' \odot e}} \\
T^n & \xrightarrow{\phi} & T
\end{array}$$

Suppose that T is a monoid in $\mathbb{C}$, and that $(\mathscr{E}, \mathscr{M})$ is a factorization system on $\mathscr{C}$ such that $\mathscr{E}$ is closed under $(-) \otimes S$ for all $S \rightarrowtail T$. Then T has a canonical grading $\mathsf{T}_{\mathscr{M}} : \mathscr{M}/\mathsf{T} \to \mathbb{C}$ by Theorem 3.11. Suppose in addition that the skew monoidal category $\mathscr{M}/\mathsf{T}$ is actually monoidal (which is the case when $\mathscr{E}$ is closed also under $S \otimes (-)$ for all $S \rightarrowtail T$). We keep these assumptions without repeating them for the rest of this section.

Our goal in the rest of this section is to show that we can assign canonical grades to algebraic operations for T. To be more precise, let $\phi : T^n \to T$ is an n-ary algebraic operation for T, and let $R_1, \ldots, R_n$ be a list of grades ($\mathscr{M}$-subobjects of T). We show how to construct a grade R' and an algebraic operation

$$\psi : \textstyle\prod_i \mathsf{T}_{\mathscr{M}}(R_i \boxdot -) \to \mathsf{T}_{\mathscr{M}}(R' \boxdot -)$$

of arity $(R_1, \ldots, R_n; R')$ for $\mathsf{T}_{\mathscr{M}}$, such that ψ grades ϕ. The grade R' is in a sense canonical (see Theorem 5.6 below), and in fact every component of ψ is in $\mathscr{E}$.

To do this, we make the following two further assumptions about $\mathscr{E}$ for the rest of the section. Firstly, we assume that $\mathscr{E}$ contains the canonical morphisms $\langle \pi_i \otimes Y \rangle_i : (\prod_i X_i) \otimes Y \to \prod_i (X_i \otimes Y)$. This is the case in particular when $\otimes$ preserves finite products on the left (because $\mathscr{E}$ contains all isomorphisms); when $\otimes$ is composition of endofunctors this is automatically true. Secondly, we assume that $\mathscr{E}$ is closed under finite products, i.e. that $\prod_i e_i : \prod_i X_i \to \prod_i Y_i$ is in $\mathscr{E}$ whenever all of the morphisms $e_i : X_i \twoheadrightarrow Y_i$ are in $\mathscr{E}$. This is the case for all of the factorization systems we consider above.

The key lemma that enables us to construct ψ is the following, which characterizes algebraic operations for the canonical grading $\mathsf{T}_{\mathscr{M}}$ of T.

Lemma 5.5. *Let $\phi : T^n \to T$ be an n-ary algebraic operation for T, and let $R_1, \ldots, R_n, R'$ be $\mathscr{M}$-subobjects of T. There is a bijection between (1) morphisms $p : \prod_i R_i \to R'$ such that*

$$\begin{array}{ccc}
\prod_i R_i & \xrightarrow{p} & R' \\
\downarrow & & \downarrow \\
T^n & \xrightarrow{\phi} & T
\end{array}$$

and (2) $(R_1, \ldots, R_n; R')$-ary algebraic operations ψ for $\mathsf{T}_{\mathscr{M}}$ that grade ϕ.

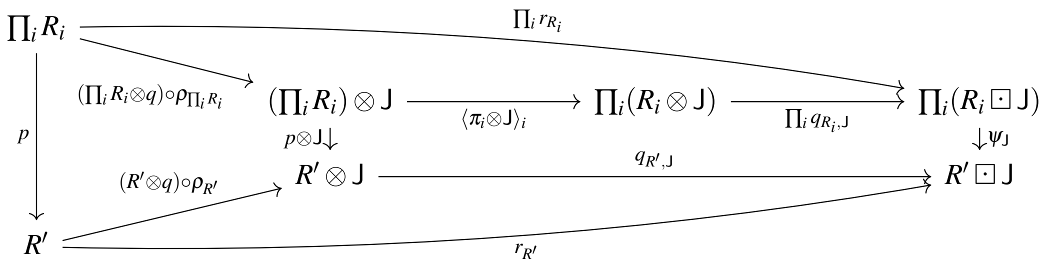
Proof. Given a morphism p as in (1), the following square commutes because ϕ is algebraic, and the square hence has a unique diagonal ψ_S. Further applications of orthogonality show that ψ is an algebraic operation. It is a grading of ϕ by definition.

$$\begin{array}{ccccc}
(\prod_i R_i) \otimes S & \xrightarrow{\langle \pi_i \otimes S \rangle_i} & \prod_i (R_i \otimes S) & \xrightarrow{\prod_i q_{R_i, S}} & \prod_i (R_i \boxdot S) \\
{\scriptstyle p \otimes S} \downarrow & & & & \downarrow \\
R' \otimes S & & \psi_S & & T^n \\
{\scriptstyle q_{R', S}} \downarrow & & & & \downarrow {\scriptstyle \phi} \\
R' \boxdot S & & \longrightarrow & & T
\end{array}$$

In the other direction, given ψ, we have a morphism p as follows; this p makes the diagram required for (1) commute because ψ is a grading of ϕ.

$$p : \prod_i R_i \xrightarrow{\prod_i r_{R_i}} \prod_i (R_i \boxdot \mathsf{J}) \xrightarrow{\psi_{\mathsf{J}}} R' \boxdot \mathsf{J} \xrightarrow{r_{R'}^{-1}} R'$$

From algebraicity of ψ it follows that this p makes the upper triangle of the above square commute and hence, by uniqueness of the diagonal, that ψ is the only grading of ϕ that induces this p. The construction of p from ψ is therefore injective. The following diagram chase shows that constructing a new p from the ψ constructed from a given p yields the same p, hence the constructions form a bijection.



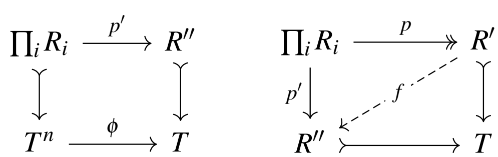
□

Now given an n-ary algebraic operation ϕ for T and a fixed tuple $R_1, \ldots, R_n$ of $\mathscr{M}$-subobjects of T, we construct the canonical R' by factorizing $\prod_i R_i \rightarrowtail T^n \xrightarrow{\phi} T$ as $\prod_i R_i \xrightarrow{p} R' \rightarrowtail T$. The preceding lemma then provides us with an $(R_1, \ldots, R_n; R')$-algebraic operation ψ for $\mathsf{T}_{\mathscr{M}}$.

Theorem 5.6. *Let $\phi : T^n \to T$ be an n-ary algebraic operation for T.*

1. *The construction above defines an $(R_1, \ldots, R_n; R')$-ary algebraic operation ψ for $\mathsf{T}_{\mathscr{M}}$, and ψ grades ϕ. Every component ψ_S is in $\mathscr{E}$.*
2. *For any $\mathscr{M}$-subobject $R'' \rightarrowtail T$ and $(R_1, \ldots, R_n; R'')$-ary algebraic operation ψ' for $\mathsf{T}_{\mathscr{M}}$, such that ψ' grades ϕ, there is a unique $f : R' \to R''$ in $\mathscr{M}/T$ such that $(f \boxdot S) \circ \psi_S = \psi'_S$ for all S.*

Proof. The first sentence of (1) is immediate from Lemma 5.5. Each ψ_S is in $\mathscr{E}$ because we have $\psi_S \circ e = e'$ for some $e, e' \in \mathscr{E}$ (this is the upper triangle in the definition of ψ_S, using the fact that p is in $\mathscr{E}$). This implies $\psi_S \in \mathscr{E}$ because $\mathscr{E}$-morphisms satisfy a two-out-of-three property. For (2), given ψ', we obtain from Lemma 5.5 a morphism $p' : \prod_i R_i \to R''$ making the diagram on the left below commute.



For a morphism $f : R' \to R''$ in $\mathscr{M}/T$, the condition that $(f \boxdot S) \circ \psi_S = \psi'_S$ for all S implies (using $S = \mathsf{J}$) that $f \circ p = p'$. The converse also holds, using orthogonality. Hence the conditions on the morphism f are equivalent to commutativity of the square on the right above. The outside of the square commutes and p is in $\mathscr{E}$, so there exists a unique f. □

Example 5.7. Consider the writer monad given by $T = M \times (-)$ from Example 3.12. Every $z \in M$ induces a unary algebraic operation $\phi_z : T \to T$, defined by $\phi_{z,X}(z', x) = (z \cdot z', x)$. When $\mathscr{M}$ is the class of componentwise injective natural transformations, the canonical $\mathscr{M}$-grading of T has subsets $\Sigma \subseteq M$

as grades, and $\mathsf{T}_{\mathscr{M}}\Sigma = \Sigma \times (-)$. Every input grade $P \subseteq M$ induces a canonical output grade $P'_z \subseteq M$ and algebraic operation $\psi_{z,\Sigma} : \mathsf{T}_{\mathscr{M}}(P \boxdot \Sigma) \Rightarrow \mathsf{T}_{\mathscr{M}}(P'_z \boxdot \Sigma)$, and these turn out to be:

$$P'_z = \{z \cdot z' \mid z' \in P\} \qquad \psi_{z,\Sigma,X}(z',x) = (z \cdot z', x)$$

Example 5.8. Let T be the list monad on **Set**. This has a binary algebraic operation $(+\!\!+) : T \times T \Rightarrow T$ that concatenates a pair of lists. As we explain in Example 4.10, subsets $\Sigma \subseteq \mathbb{N}$ provide a canonical grading of T. If P_1, P_2 are subsets of $\mathbb{N}$, then the grade we construct for the algebraic operation $(+\!\!+)$ as above is $P' = \{n_1 + n_2 \mid n_1 \in P_1, n_2 \in P_2\}$, and the algebraic operation for $\mathsf{T}_{\mathscr{M}}$ is the natural transformation $\mathsf{T}_{\mathscr{M}}(P_1 \boxdot -) \times \mathsf{T}_{\mathscr{M}}(P_2 \boxdot -) \Rightarrow \mathsf{T}_{\mathscr{M}}(P' \boxdot -)$ that maps (xs_1, xs_2) to $xs_1 +\!\!+ xs_2$.

6 Conclusion and future work

We have demonstrated that factorization systems provide a unifying framework for the grading of monads by subfunctors, in fact, monoids with subobjects. Skew monoidal categories turn out to be a more robust setting for this than monoidal categories, which means, among other things, that this framework will be directly applicable also to relative monads.

The abstract framework is pleasingly elegant, but for applications we would like obtain a stronger intuition for its reach. We intend to explore this first by working out the canonical gradings with (strong) subfunctors of further standard example (strong) monads from programming semantics, for the factorization systems considered in this paper and possibly others. Indeed, the examples may point to further factorization systems of interest. The outcomes of this exploration will hopefully lead to some new heuristics for the construction of graded monads for applications such as type-and-effect systems.

Programming semantics applications also suggest trying grading with subfunctors on (strong) lax monoidal functors ("applicative functors") and (strong) monads in **Prof** ("arrows"). Comonads can be graded with quotient functors.

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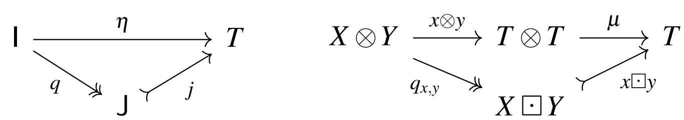
A Proof of Theorem 3.9

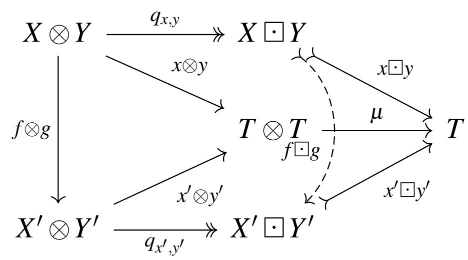
Given a monoidal category $(\mathscr{C}, \mathsf{I}, \otimes, \lambda, \rho, \alpha)$ with a monoid object (T, η, μ) and an orthogonal factorization system $(\mathscr{E}, \mathscr{M})$. We assume that $\mathscr{E}$ is closed under $(-) \otimes X$ for all $(X, x) \in \mathscr{M}/T$.

Our aim is to show $\mathscr{M}/T$ carries a left-skew monoidal category structure $((\mathsf{J}, j), \boxdot, \ell, r, a)$.

The unit (J, j) and tensor $(X \boxdot Y, x \boxdot y)$ of two objects (X, x), (Y, y) are defined as the factorizations shown in the diagrams below.

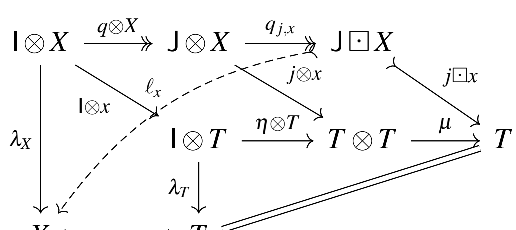


The functorial action of $\boxdot$ on two morphisms $f : (X, x) \to (X', x')$ and $g : (Y, y) \to (Y', y')$ is a morphism $f \boxdot g : (X \boxdot Y, x \boxdot y) \to (X' \boxdot Y', x' \boxdot y')$ defined as the diagonal fill-in of the commuting square below.

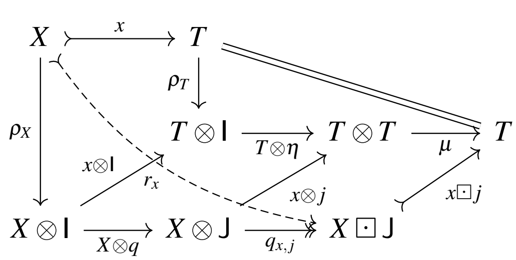


The left unitor ℓ and associator a are also defined as the diagonal fill-ins for suitable commuting squares. The right unitor is just a composition of morphisms.

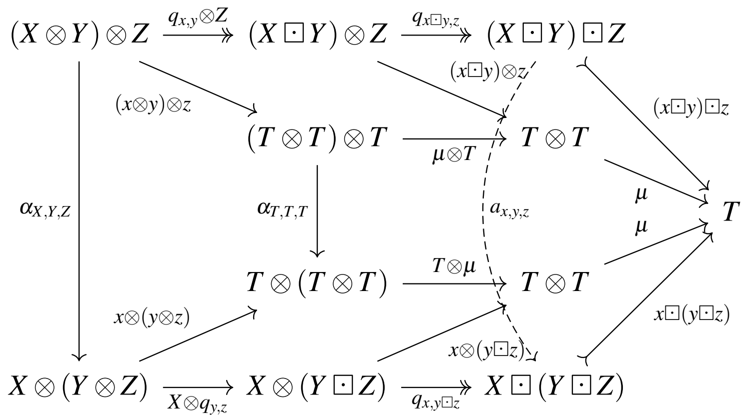
Definition of ℓ:



Definition of r:



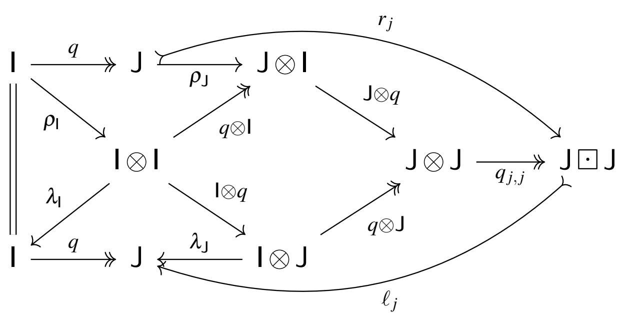
Definition of a:



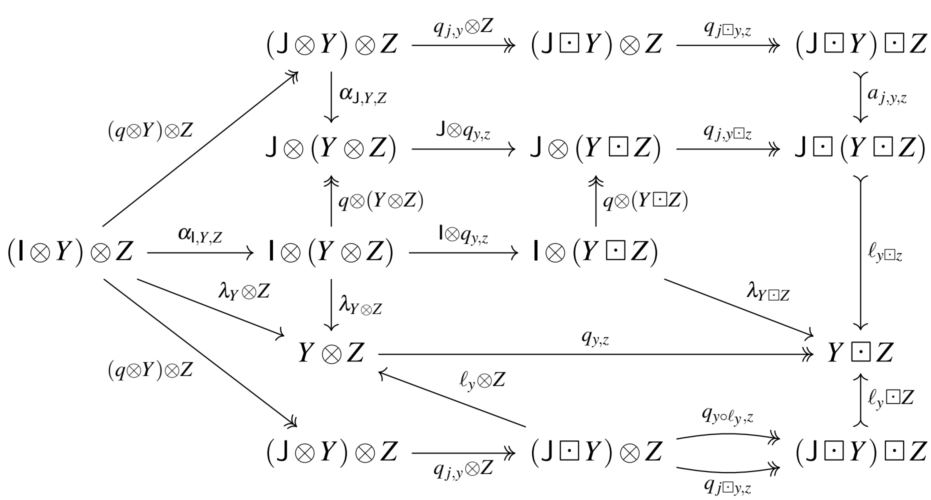
The proofs of functoriality of $\boxdot$ and naturality of ℓ, r and a are easy and omitted.

The equations (m1)–(m5) for ℓ, r, a are each proved from the respective equations of λ, ρ, α using the properties of $\boxdot$, ℓ, r, a arising from their construction (the two triangles that the fill-in breaks the square into). For each equation $lhs = rhs$, the two sides lhs and rhs are both shown to be the diagonal fill-in of a square of the form $s \circ e = s' \circ f$ where e is a suitable $\mathscr{E}$-morphism and s and s' are the common domain resp. codomain of lhs and rhs as morphisms in $\mathscr{M}/T$. Below are the diagram chases for the triangles $lhs \circ e = f = rhs \circ e$; the triangles $s' \circ lhs = s = s' \circ rhs$ are straightforward.

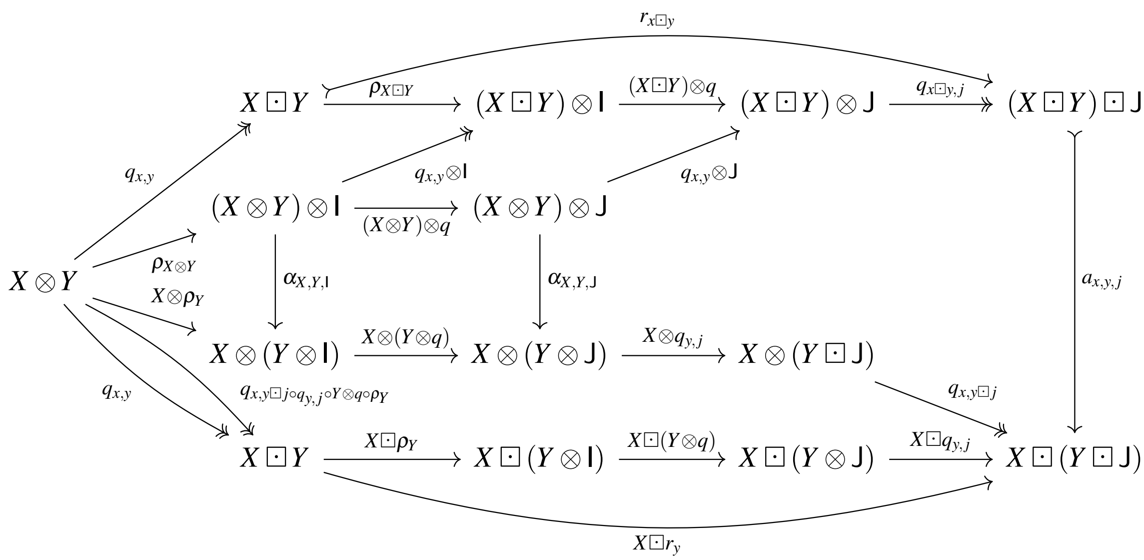
Proof of (m1):



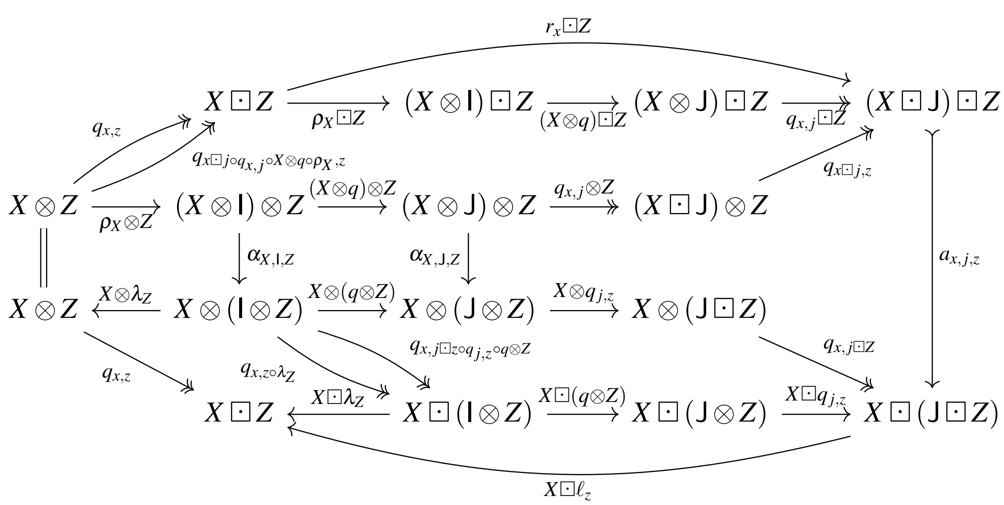
Proof of (m2):



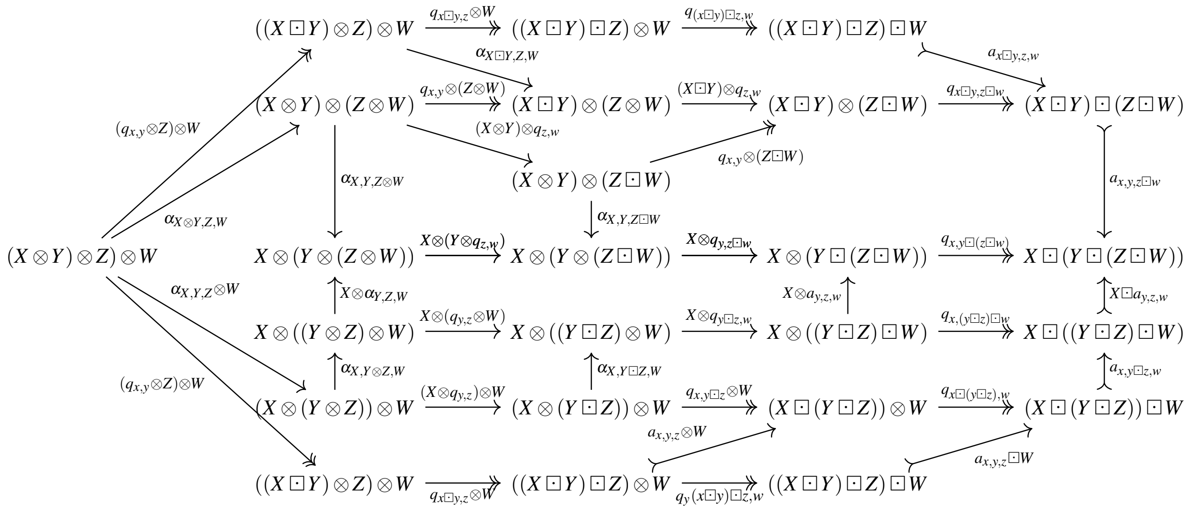
Proof of (m3):



Proof of (m4):



Proof of (m5):


((X ⊡ Y) ⊗ Z) ⊗ W
((X ⊡ Y) ⊡ Z) ⊗ W
((X ⊡ Y) ⊡ Z) ⊡ W
(X ⊗ Y) ⊗ (Z ⊗ W)
(X ⊡ Y) ⊗ (Z ⊗ W)
(X ⊡ Y) ⊗ (Z ⊡ W)
(X ⊡ Y) ⊡ (Z ⊡ W)
(X ⊗ Y) ⊗ (Z ⊡ W)
(X ⊗ Y) ⊗ Z) ⊗ W
X ⊗ (Y ⊗ (Z ⊗ W))
X ⊗ (Y ⊗ (Z ⊡ W))
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